Dynamic Price Competition:
Theory and Evidence from Airline Markets*

Ali Hortaçsu — University of Chicago and NBER
Aniko Öry — Yale University
Kevin R. Williams† — Yale University and NBER

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Abstract

We estimate welfare effects of dynamic price competition in the airline industry. To do so, we introduce a general dynamic pricing game where sellers are endowed with finite capacities and face uncertain demands toward a sales deadline. We establish sufficient conditions for equilibrium existence and uniqueness, and for convergence to a system of differential equations. With the equilibrium characterization and comprehensive pricing and bookings data for competing airlines, we estimate that dynamic pricing results in higher output but lower welfare than under uniform pricing.

*The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research. We thank the anonymous airline for giving us access to the data used in this study. Under the agreement with the authors, the airline had "the right to delete any trade secret, proprietary, or Confidential Information" supplied by the airline. We agreed to take comments in good faith regarding statements that would lead a reader to identify the airline and damage the airline's reputation. All authors have no material financial relationships with entities related to this research. We thank Jose Betancourt for his excellent research assistance. We thank seminar participants at NYU, Yale, and SICS 2022 for comments.

†Emails: hortacsu@gmail.com, aniko.oery@yale.edu, kevin.williams@yale.edu

1 Introduction

Dynamic pricing is commonly used by firms selling fixed inventory by a set deadline. Examples range from seats on airlines and trains, tickets for entertainment events, to reservations for cruises, and inventory in retailing. In these markets, capacity influences prices in important ways. First, prices adjust as the opportunity cost of selling changes with scarcity—the value of a capacity unit depends on the ability to sell it in the future. Second, demand may change over time which can provide an incentive to hold inventory for certain customers. In all of the aforementioned examples, firms also face competition. Therefore, there exists a third force in that the opportunity cost of selling also depends on other firms' inventories as they affect future prices. An open theoretical and empirical question is how scarcity among competing firms affects dynamic prices and welfare.

In this paper we estimate the welfare effects of dynamic price competition in the airline industry using new theoretical insights and granular data on competing airlines. We introduce a general dynamic pricing game and establish sufficient conditions for equilibrium existence and uniqueness, and for convergence to a system of differential equations for an arbitrary number of firms and products. Our theoretical results show how little intuition from the well-studied single-firm setting carries over to markets where firms compete because of sellers' incentives to soften future price competition. For example, a firm may fire-sale units even if it has the smallest inventory in the industry in order to increase future prices. Or, a firm with excess capacity may charge high prices in order to get a competitor to sell out early. We undercover these strategic incentives in the airline industry by first estimating a model of air travel demand using daily pricing and bookings data of competing airlines in U.S. oligopoly markets. With the continuous-time equilibrium characterization of the model, we estimate that dynamic pricing expands output, increases firm revenues, lowers consumer surplus, and decreases total welfare compared to uniform pricing. Our results contrast recent empirical studies, largely focused on single-firm settings, where dynamic pricing is found to increase welfare (Hendel and Nevo, 2013; Castillo, 2020; Williams, 2022).

Our dynamic pricing game extends earlier single-firm frameworks (Gallego and Van Ryzin, 1994; Zhao and Zheng, 2000; Talluri and Van Ryzin, 2004) to oligopoly.[1] In the main text we focus on a duopoly where each firm offers a single product. In the appendix, we extend our results to an arbitrary number of firms, each offering an arbitrary number of products. Each firm is exogenously endowed with limited initial capacity that must be sold by a deadline.[2] After the deadline has passed, unsold capacities are scrapped with zero value. Firms are not allowed to oversell. Products are imperfect substitutes and satisfy general regularity conditions. Consumers arrive randomly according to time-varying arrival rates with time-varying preferences. Each consumer is short-lived and decides whether to purchase an available product or select an outside option. Our demand assumptions are motivated by recent empirical evidence (Hortaçsu et al., 2021b). In every period, firms simultaneously choose prices after observing remaining capacities for all products; demand is realized, capacity constraints are updated, and the process repeats until the perishability date or until all products are sold out. We call this game the *benchmark model*.

Our model produces a rich set of equilibrium strategies because competitor prices affect both current demand and opportunity costs of remaining capacity. This can create incentives to offer fire sales as in Dilme and Li (2019), where a single firm competes with its future self for forward-looking buyers.[3] However, a firm might also want to charge a high price in order to drive the competitor to sell out as in Martínez-de Albéniz and Talluri (2011), where firms offer perfect substitutes.[4]

We show that the incentive to soften future competition puts upward pressure on a firm's price today depending on whether a sale of the competitor increases the firm's expected future profits. We introduce the concept of the "competitor scarcity effect" defined as the loss in continuation profit when the competitor sells. Simulations suggest that the

[1] There exists a large literature on dynamic price competition in various other settings, e.g., Maskin and Tirole (1988); Dana (1999); Bergemann and Välimäki (2006); Sweeting et al. (2020).

[2] See Dana and Williams (2022) for a related model that endogenizes the capacity decision.

[3] Board and Skrzypacz (2016); Gershkov et al. (2018) consider forward-looking buyers when the firm can fully commit to a selling mechanism and hence, resist the temptation to fire-sale.

[4] Similar incentives also arise in Edgeworth cycles (Dudey, 1992).

competitor scarcity effect is typically negative, i.e., a firm benefits from shifting demand to its rival, so that each firm generally prefers that its rival sells out early. It can then raise prices later on. The "own-product scarcity effect"—defined as the loss in continuation value if an own product sells—is typically positive which puts upward pressure on the own price. However, we show that all scarcity effects can be positive or negative and can cause the competitor's price to become a strategic substitute of the firm's price. We establish a link between scarcity effects and remaining capacity; we formally show that the scarcity effects increase the most in size when the firm with the lowest capacity sells. Competition is fiercest when firms have the same number of units remaining.

The presence of competitor scarcity effects implies that firms' payoffs in the stage game are generally neither supermodular nor log-supermodular (Milgrom and Roberts, 1990). They are also not of the form studied in either Caplin and Nalebuff (1991) or Nocke and Schutz (2018). To make further progress, we derive sufficient conditions for existence and uniqueness of equilibria of the stage game using a theorem in Kellogg (1976). Although we show that even simple parametrizations of the model using logit demand may yield multiple equilibria and price jumps, we prove that close to the deadline, our sufficient conditions for existence and uniqueness are always satisfied for commonly used demand systems. These conditions also ensure that the unique equilibrium price paths in the continuous-time limit of a discrete-time game satisfy a system of differential equations.

We use our theoretical framework and comprehensive data on competing airlines to quantify the welfare effects of dynamic price competition in the airline industry. This industry has been noted for significant price dispersion within and across routes (Borenstein and Rose, 1994; Stavins, 2001; Gerardi and Shapiro, 2009; Berry and Jia, 2010; Puller et al., 2012; Sengupta and Wiggins, 2014; Siegert and Ulbricht, 2020). We use new data sources that provide not only prices, but also all bookings (specifically, booking counts) for all competing carriers on a given route. The booking counts include tickets purchased directly with the airline and all other sources, e.g., online travel agencies, for every flight.

We estimate a Poisson demand model, where aggregate demand uncertainty is captured through Poisson arrivals, and preferences are modeled through discrete choice nested logit demand. We use search data for one airline to inform arrival process parameters that are then scaled up to account for unobserved searches, e.g., via online travel agencies or a competitor's website. In total, we estimate demand for 58 duopoly routes. We find significant variation in willingness to pay across routes and across days from departure for a given route. In general, demand becomes more inelastic as the departure date approaches. Average own-price elasticities are -1.4.

With the demand estimates, we simulate equilibrium market outcomes using the differential equation characterization. This allows us to recover the own/competitor scarcity effects and firm strategies for all potential states—some games (route-departure dates) feature over 131 million potential states. We find that overwhelmingly (but not all) of the realized stage games are of strategic complements.

We compare market outcomes of dynamic pricing to uniform pricing where each firm commits to a single price for each flight over time. We find the opposite welfare effect compared to earlier analyses, including Hendel and Nevo (2013) in retailing, Castillo (2020) in ride-share, and Williams (2022) for single-carrier airline markets, in that dynamic pricing expands output but lowers total welfare compared to uniform pricing. This occurs because dynamic pricing softens price competition toward the departure date, despite featuring lower average prices on average. Our estimates suggest that uniform pricing would increase total welfare by 2.2% but lower quantity sold by 6.4%.

We also investigate two pricing heuristics that mimic some industry pricing practices.[5] The algorithms differ from recent work in economics that study reinforcement algorithms (Calvano et al., 2020; Asker et al., 2021; Leisten, 2021; Hansen et al., 2021) in that the airline does not learn over time (within a departure date). We find that heuristics lead to ambiguous effects on firm revenues but result in higher welfare than under the benchmark model. That is, the full-information, dynamic pricing game results in the lowest welfare among all counterfactuals.

[5] We observe how one airline incorporates competition into their models via their documentation.

2 Model of Dynamic Price Competition

We begin by detailing the demand assumptions that we use in our analysis in Section 2.1. Our exposition of demand is for an arbitrary number of products. In Section 2.2 we introduce supply-side notation by examining the single firm case. We then introduce a duopoly pricing game with two products in Section 2.3 which we analyze in Section 3. In Appendix A, we generalize all results to a pricing game with arbitrary number of firms and products.

2.1 Demand Model

We consider an economy with a set of products denoted by $\mathscr{J} := \{1,\ldots,J\}$. Products are imperfect substitutes and must be scrapped with zero value at a deadline $T > 0$. We analyze a discrete-time environment with periods $t \in \{0,\Delta,\ldots,T-\Delta\}$, $\Delta > 0$, and later study the continuous-time approximation as $\Delta \to 0$. In every period, a consumer arrives with probability $\Delta\lambda_t$. Therefore, each consumer can be indexed by the time t of her arrival.

If all products are available, then given a vector of prices $\mathbf{p} = (p_j)_{j\in\mathscr{J}}$, consumer t purchases product j with probability $s_j(\mathbf{p}\,;\,\boldsymbol{\theta}_t,\mathscr{J})$, where $\boldsymbol{\theta}_t \in \Theta \subset \mathbb{R}^n$ is a vector of $n \geq 1$ parameters that are smooth and deterministic in time t. We impose the following regularity conditions on the demand system.

Assumption 1. *For all $\boldsymbol{\theta} \in \Theta$ and $\mathbf{p} \in \mathbb{R}^{\mathscr{J}}$, the following hold:*

i) Convergence for infinite prices: *For any j, $\lim_{p_j\to\infty} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) = 0$. For any subset $\mathscr{A} \subset \mathscr{J}$ and $j \in \mathscr{A}$, the limit*[6]

$$s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) := \lim_{\substack{p_{j'}\to\infty \\ j'\notin\mathscr{A}}} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) \in [0,1]$$

exists, where $p_{j'} = p^{\mathscr{A}}_{j'}$ for all $j' \in \mathscr{A}$, $\mathbf{p}^{\mathscr{A}} \in \mathbb{R}^{\mathscr{A}}$;

ii) Products are imperfect substitutes: *For all j, $s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})$ is strictly decreasing in p_j and strictly increasing in $p_{j'}$, $j' \neq j$;*

[6] The limit takes all prices of products $j \notin \mathscr{A}$ to infinity where the order does not matter.

iii) Differentiability and diagonally dominant Jacobi matrix: *For all subsets* $\mathscr{A} \subset \mathscr{J}$ *and* $j \in \mathscr{A}$, $s_j(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})$ *is smooth in* $\boldsymbol{\theta}$ *and* $\mathbf{p}^{\mathscr{A}}$, *and*

$$\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})\right| > \sum_{j' \in \mathscr{A} \setminus \{j\}} \left|\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})\right|; \tag{1}$$

Furthermore, for any subset $\mathscr{A} \subset \mathscr{J}$ *there exists a* $C > 0$ *such that for all* $\mathbf{p}^{\mathscr{A}}$

$$|s_j(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})| < C \cdot \left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})\right| - \sum_{j' \in \mathscr{A} \setminus \{j\}} \left|\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p}^{\mathscr{A}}; \boldsymbol{\theta}, \mathscr{A})\right| \right). \tag{2}$$

Assumption 1-i) ensures that the demand system is well-defined when products sell out. The first condition in Assumption 1-iii) (Equation 1) ensures that the Jacobi matrix $D_{\mathbf{p}} s(\cdot)$ is non-singular by the Levy-Desplanques Theorem (see e.g. Theorem 6.1.10. in Horn and Johnson (2012)). This condition intuitively means that a price change of product j should impact demand of product j more than it impacts the sum of demand of all other products. The second condition in Assumption 1-iii) (Equation 2) ensures that the demand for each product is bounded away from 1 and the differential impact of price changes is large relative to demand. We will use this condition to establish that optimal prices for a single firm and best responses for each firm in a oligopoly game are uniformly bounded given any parameter value $\boldsymbol{\theta}$ and set of available products $\mathscr{A}$.

Given Assumption 1, and denoting $\mathbf{s}$ to be the vector of s_js, we can define for any $\boldsymbol{\theta}$, $\mathscr{A}$, and finite price vector $\mathbf{p} \in \mathbb{R}^{\mathscr{A}}$ the vector of inverse quasi own-price elasticities of demand as

$$\hat{\boldsymbol{\epsilon}}(\mathbf{p}; \boldsymbol{\theta}, \mathscr{A}) := \left(D_{\mathbf{p}} \boldsymbol{s}(\mathbf{p}; \boldsymbol{\theta}, \mathscr{A})\right)^{-1} \boldsymbol{s}(\mathbf{p}; \boldsymbol{\theta}, \mathscr{A}).$$

Assumption 2 details the assumption that we place on demand elasticities.

Assumption 2. *The vector of inverse quasi own-price elasticities* $\hat{\boldsymbol{\epsilon}}(\mathbf{p}; \boldsymbol{\theta}, \mathscr{A})$ *satisfies* $\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}}(\mathbf{p}; \boldsymbol{\theta}, \mathscr{A}) - I\right) \neq 0$ *for all* $\mathbf{p} \in \mathbb{R}^{\mathscr{A}}$, $\boldsymbol{\theta} \in \Theta$ *and,* $\mathscr{A} \subset \mathscr{J}$, *where* I *is the identity matrix.*

Recall that Assumption 1 guarantees that $\max_{\mathbf{p}\in\mathbb{R}^{\mathscr{A}}} \mathbf{s}(\mathbf{p})^\intercal(\mathbf{p}-\mathbf{c})$ has an interior solution. Assumption 2 guarantees that the system of first-order conditions of this problem has a unique solution. Together, these assumptions replace the assumption of log-concavity commonly made in single-product, single-firm setting.

We now omit the conditioning arguments $\boldsymbol{\theta}$ and $\mathscr{A}$ in demand and demand elasticities whenever the meaning is unambiguous. When the time index is relevant, we write $s_{j,t}(\mathbf{p}) := s_j(\mathbf{p};\boldsymbol{\theta}_t, \mathscr{A}_t)$. Further, we let the probability of choosing the outside option be equal to $s_{0,t}(\mathbf{p}) := 1 - \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})$.

2.1.1 Parametric Demand Models

We illustrate theoretical insights with a simple logit demand specification, i.e.,

$$s_{j,t}(\mathbf{p}) = \frac{\exp\left\{\frac{\delta_j-\alpha_t p_j}{\rho}\right\}}{1+\sum_{j'\in\mathscr{A}_t}\exp\left\{\frac{\delta_{j'}-\alpha_t p_{j'}}{\rho}\right\}}. \tag{3}$$

We set $\boldsymbol{\theta}_t = \alpha_t$ so that α_t/ρ is the time-variant marginal utility to income, and $\rho > 0$ is a scaling factor. The parameter δ_j/ρ is the product-specific value of product j. Note that when $\rho \to 0$, competition collapses to standard Bertrand. As $\rho \to \infty$, products become perfectly differentiated. In our empirical analysis, we consider the more flexible nested logit demand model. Both classic logit and nested logit demand functions satisfy Assumptions 1 and 2 (see Appendix C).

2.2 Single Firm Model

We first discuss a single firm, multi-product dynamic pricing model with two goals in mind. The first is to introduce supply-side notation that we carry over to the competitive model. The second is to showcase that the single-firm problem is well behaved and exhibits nice properties. All of them fail in the oligopoly model.

A single firm M offers J products for sale with an initial inventory $K_{j,0} \in \mathbb{N}$ of each product j. We do not model the initial capacity choice. Let $\mathbf{K}_t = (K_{j,t})_{j\in\mathcal{J}}$ denote the capacity vector at time t. The firm's continuation payoff at time $t \leq T$, given capacity vector $\mathbf{K}$, satisfies the dynamic program

$$\Pi_{M,t}(\mathbf{K};\Delta) = \max_{\mathbf{p}} \Delta\lambda_t \sum_{j\in\mathcal{J}} s_{j,t}(\mathbf{p})\Big(p_j + \Pi_{M,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)\Big) + \Big(1-\Delta\lambda_t \sum_{j\in\mathcal{J}} s_{j,t}(\mathbf{p})\Big)\Pi_{M,t+\Delta}(\mathbf{K};\Delta),$$

where $\mathbf{e}_j \in \mathbb{N}^{\mathcal{J}}$ is a vector of zeros with a one in the jth position. The firm faces three boundary conditions: (i) $\Pi_{M,T+\Delta}(\cdot;\Delta) = 0$, (ii) $\Pi_{M,t}(\mathbf{0};\Delta) = 0$ for all t, where $\mathbf{0}$ is a vector of zeros, and (iii) $\Pi_{M,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathcal{J}$. These boundary conditions are simply stating that any remaining capacity is scrapped with zero value after the deadline T, and that the firm cannot oversell. Note that the prices in period t do not directly affect the continuation values in period $t+\Delta$. Hence, the optimal price in each period solves a static maximization problem given the continuation payoffs. We denote this static profit-maximizing price vector parameterized by $\omega_j := \Pi_{M,t}(\mathbf{K};\Delta) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j;\Delta)$, commonly referred to as the *opportunity cost of selling product* j, by

$$\mathbf{p}_M(\boldsymbol{\omega}) := \arg\max_{\mathbf{p}} \sum_{j\in\mathcal{J}} s_j(\mathbf{p})\big(p_j - \omega_j\big),$$

where $\boldsymbol{\omega} = (\omega_j)_{j\in\mathcal{J}}$.[7] By Lemma 2 in Konovalov and Sándor (2010), Assumption 2 immediately implies that there is a unique optimal price vector which is continuous in $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. Then, by Lemma 5 in the Appendix, the continuous-time limit of this dynamic program exists, is unique, and solves the differential equation specified in the following lemma. The lemma formalizes that the loss in continuation profit if no sale occurs is given by the forgone expected flow revenue $\lambda_t \max_{\mathbf{p}} \sum_{j\in\mathcal{J}} s_{j,t}(\mathbf{p})\big(p_j - (\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j))\big)$.

Lemma 1. *Let Assumptions 1 and 2 hold. Then,* $\Pi_{M,t}(\mathbf{K};\Delta)$ *converges uniformly to* $\Pi_{M,t}(\mathbf{K})$

[7] Note that strictly speaking, the opportunity cost of selling product j is given by $\omega_j - \sum_{j'\neq j} \frac{s'_j(\mathbf{p})}{1-s_j(\mathbf{p})}\omega_{j'}$ as by selling product j, the firm forgoes the opportunity to sell any other product to the customer.

as $\Delta \to 0$*, which satisfies*

$$\dot{\Pi}_{M,t}(\mathbf{K}) = -\lambda_t \max_{\mathbf{p}} \sum_{j \in \mathscr{J}} s_{j,t}(\mathbf{p}) \Big(p_j - \big(\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K} - \mathbf{e}_j) \big) \Big)$$

with boundary conditions (i) $\Pi_{M,T}(\mathbf{K}) = 0$ *for all* $\mathbf{K}$*, (ii)* $\Pi_{M,t}(\mathbf{0}) = 0$ *for all* t*, and* $\Pi_{M,t}(\mathbf{K}) = -\infty$ *if* $K_j < 0$ *for a* $j \in \mathscr{J}$*.*

Given a capacity vector $\mathbf{K}$, corresponding available products $\mathscr{A} = \{j : K_j \neq 0\}$, and the vector of opportunity costs $\boldsymbol{\omega}_{M,t}(\mathbf{K})$ of products $j \in \mathscr{A}$, the first-order condition for profit-maximizing prices $\mathbf{p}_{M,t}(\mathbf{K}) \in \mathbb{R}^{\mathscr{A}}$ can be written in matrix form,

$$\mathbf{p}_{M,t}(\mathbf{K}) = \underbrace{\boldsymbol{\omega}_{M,t}(\mathbf{K})}_{\text{opportunity costs}} - \underbrace{\left(D_{\mathbf{p}}\mathbf{s}_t(\mathbf{p}_{M,t}(\mathbf{K}))\right)^{-1}\mathbf{s}_t(\mathbf{p}_{M,t}(\mathbf{K}))}_{\substack{= \hat{\epsilon}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) \\ \text{inverse quasi own-price elasticities}}} \quad . \tag{4}$$

Hence, the pricing policy $\mathbf{p}_{M,t}(\mathbf{K})$ is continuous in time and well behaved. The evolution of the price vector $\mathbf{p}_{M,t}(\mathbf{K}_t)$ is then governed by the evolution of the random variable representing the opportunity costs and quasi-price elasticities of demand. The following proposition summarizes well-known properties of an optimal control problem, including monotonicity and concavity of the value function in the capacity vector. We also derive properties of the stochastic process governing the opportunity costs $\omega_{j,t}(\mathbf{K}_t)$.

Proposition 1. *The solution to the continuous-time single-firm revenue maximization problem in Lemma 1 satisfies the following:*

i) $\Pi_{M,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and increasing in* K_j *for all* $j \in \mathscr{J}$ *and* $t < T$*;*

ii) $\omega_{j,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and decreasing in* K_j *for all* j *and* $t < T$*;*

iii) The stochastic process $\omega_{j,t}(\mathbf{K}_t)$ *is a submartingale.*

Statements i) and ii) of Proposition 1 simply state that more capacity and more time remaining increase continuation profits, that every additional unit of capacity increases profits by less (concavity of profits in capacity), and that opportunity costs are increasing towards the deadline if $\mathbf{K}$ is held fixed. Statement (iii) implies that on average opportunity costs are increasing, i.e., prices are increasing if $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$ by (4). This formal result has been shown in simulations, e.g., in McAfee and Te Velde (2006), where close to the deadline, observed prices decrease since the infinite prices of sold out products are ignored.

2.3 Duopoly Model

We introduce a duopoly pricing game with two firms $f \in \{1,2\}$. Each firm controls exactly one product, i.e., $\mathscr{J} = \{1,2\}$. Therefore, we set $j = f$ and use the subscript f to denote both the firm and product of interest. We generalize the results in this section to multiple firms with multiple products in Appendix A. Our exposition here focuses on the duopoly case with two products since this case is sufficient to highlight the key forces relevant for our analysis. Each firm f is initially endowed with $K_{f,0}$ units of its own product. In every period, firms simultaneously set prices $p_{f,t}$, and then a consumer arrives with probability $\Delta\lambda_t$. If a consumer arrives, she buys a product from firm f with probability $s_{f,t}(p_{1,t}, p_{2,t})$.

As in the single firm case, the payoff-relevant state is given by the vector of inventories $\mathbf{K} := (K_1, K_2)$ at time t. We study Markov perfect equilibria in which each firm's strategy is measurable with respect to (K_1, K_2, t). We denote a Markov strategy of firm f by $p_{f,t}(\mathbf{K})$. Given equilibrium price vectors $\mathbf{p}_t^*(\mathbf{K}) := (p_{1,t}^*(\mathbf{K}), p_{2,t}^*(\mathbf{K}))$, firm f's value function satisfies[8]

$$\begin{aligned}\Pi_{f,t}(\mathbf{K};\Delta) = \Delta\lambda_t \Bigg(& \underbrace{s_{f,t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\left(p_{f,t}^*(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_f;\Delta)\right)}_{\text{revenue from own sale}} + \\ & \underbrace{s_{f',t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{f'};\Delta)}_{\text{continuation value if } f' \text{ sells}}\Bigg) + \underbrace{\Bigg(1 - \Delta\lambda_t \sum_{h=\{1,2\}} s_{h,t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\Bigg)}_{\text{probability of no purchase}} \cdot \Pi_{f,t+\Delta}(\mathbf{K};\Delta),\end{aligned} \tag{5}$$

where we denote the competitor by $f' \neq f$. The boundary conditions are analogous to

[8] Formally, equilibrium prices are a function of Δ, which we omit in the main text for readability.

the single-firm case: (i) $\Pi_{f,T+\Delta}(\mathbf{K};\Delta) = 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) = 0$ if $K_f = 0$, (iii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_f < 0$, (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{f'};\Delta) = \Pi_{f,t}(\mathbf{K};\Delta)$ if $K_{f'} = 0$, $K_f \geq 0$.

Similar to the single-firm setup, the period-t price vector does not impact the continuation payoffs in period $t+\Delta$. Hence, $\mathbf{p}_t^*(\mathbf{K})$ is an equilibrium of a stage game in which firm f's payoff is given by $\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta)$. In order to describe this stage game, we denote for each firm $f \in \{1,2\}$ the change in continuation profit if product $h \in \{1,2\}$ by

$$\omega_{h,t}^{f}(\mathbf{K}) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_h;\Delta),$$

which we call the *scarcity effect* of product h on firm f. We refer to $\omega_{f,t}^{f}$ as the *own-product scarcity effect* and $\omega_{f',t}^{f}$, $f' \neq f$ as the *competitor scarcity effect*. We set $\omega_{f',t}^{f} := 0$ if $K_{f'} = 0$.[9] Then, the stage game is parameterized by the matrix of scarcity effects

$$\Omega_t(\mathbf{K}) = \begin{pmatrix} \omega_{1,t}^{1}(\mathbf{K}) & \omega_{2,t}^{1}(\mathbf{K}) \\ \omega_{1,t}^{2}(\mathbf{K}) & \omega_{2,t}^{2}(\mathbf{K}) \end{pmatrix},$$

where by Equation (5), firm f's flow payoff is equal to

$$\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta) = \Delta\lambda_t \left(s_{f,t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\left(p_{f,t}^*(\mathbf{K}) - \omega_{f,t}^{f}(\mathbf{K})\right) - s_{f',t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\omega_{f',t}^{f}(\mathbf{K})\right).$$

Hence, in the stage game, firms simultaneously choose prices and receive payoffs

$$s_{f,t}(\mathbf{p})(p_f - \omega_{f,t}^{f}(\mathbf{K})) - s_{f',t}(\mathbf{p})\omega_{f',t}^{f}(\mathbf{K}), \qquad f' \neq f.$$

Intuitively, the firm incurs an opportunity cost of selling its own product as in the single-firm setting, but future prices are also affected by the future degree of competition. For example, firm f benefits from a sale of the competitor if $\omega_{f'}^{f} < 0$. This provides the firm an incentive to shift demand to the competitor. The stage game can have different strategic properties depending on the size and sign of the scarcity effects. We introduce the following

[9]We do not call the ωs opportunity costs for the same reason as discussed in Footnote 7.

terminology.

Definition 1. We say that a *competitor's sale intensifies competition* in a state $(\mathbf{K}, t)$ if $\omega^f_{f',t}(\mathbf{K}) > 0$ and that a *competitor's sale softens competition* in a state $(\mathbf{K}, t)$ if $\omega^f_{f',t}(\mathbf{K}) < 0$.

For a stage game with $\omega^f_{f',t} \neq 0$, we cannot apply results from Caplin and Nalebuff (1991) or Nocke and Schutz (2018). Payoffs are also neither super-modular nor log-supermodular (Milgrom and Roberts, 1990), and the stage game is also not a potential game. In the next section, we derive conditions on the stage game that guarantee uniqueness of equilibrium outcomes to show in how far Lemma 1 generalizes to a duopoly.

3 Analysis of the Duopoly Model

In this section, we derive theoretical properties of the dynamic pricing game. We start with an analysis of uniqueness and continuity of stage game equilibria, which allows us to generalize Lemma 1. We also provide additional theoretical insights on competition, the role of capacity, and pricing dynamics.

3.1 Equilibrium Existence, Uniqueness, and Continuity

3.1.1 Sufficient Condition for Equilibrium Uniqueness in the Stage Game

We consider the stage game for an arbitrary matrix of opportunity costs Ω. We drop the time index and capacity argument in all expressions temporarily. Our first result presents sufficient conditions for existence and uniqueness of an equilibrium of the stage game. Recall that the best responses of both firms are uniformly bounded by Assumption 1-(iii) and hence, must satisfy a first-order condition. We can write the first-order condition of firm f's profit maximization problem as

$$g_f(\mathbf{p}) = p_f,$$

where

$$g_f(\mathbf{p}) := \underbrace{\omega_f^f + \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})}\omega_{f'}^f}_{\substack{\text{net opportunity cost} \\ \text{of selling}}} - \underbrace{s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1}}_{\substack{\text{inverse quasi} \\ \text{own-price elasticity}}}. \tag{6}$$

By Kellogg (1976),[10] the following assumption then guarantees that there is a unique solution to this system of equations.

Assumption 3. *Suppose the following two conditions hold:*

i) $\frac{\partial g_f}{\partial p_f}(\mathbf{p}) - 1 \neq 0$ *for all* $\mathbf{p}$ *and* $f = 1, 2$*;*

ii) $\det\left(D_{\mathbf{p}}\big(\mathbf{g}(\mathbf{p})\big) - I\right) \neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p}) := (g_1(\mathbf{p}), g_2(\mathbf{p}))$*.*

To better understand Assumption 3, first note that with a single firm, the assumption guarantees that the first-order condition of the firm is either increasing or decreasing everywhere in its price. Assumption 3-(i) is always satisfied for demand functions that are log-concave in each dimension. Mathematically, Assumption 3-(ii) is related to Assumption 2, but the inverse quasi-own price elasticity is replaced by the function $\mathbf{g}(\mathbf{p})$. If the competitor scarcity effect is zero, one can see from Equation (6) that Assumption 2 implies Assumption 3. If the competitor scarcity effect is not zero, the first-order condition is more complex than in the single-firm setting since the net opportunity cost of selling depends on the ratio of derivatives of the demand of the two firms.

Lemma 2. *Let Assumptions 1, 2 and 3 hold. Then, the stage game admits a unique equilibrium.*

Note that Lemma 2 establishes uniqueness and existence simultaneously. Under the commonly made assumption of independence of irrelevant alternatives (IIA) that is satisfied by a classic logit demand specification, existence of an equilibrium is always guaranteed. Finally, note that the conditions in Assumption 3 depend on Ω, i.e., they might not guarantee

[10] See Lemma 2 in Konovalov and Sándor (2010).

uniqueness for arbitrary stage games. In the next subsection we provide an example for Ω that yield multiple equilibria.

3.1.2 Continuity of Equilibrium Prices in Scarcity Effect Matrix Ω

Next, we study the stage game parameterized by scarcity effects Ω and demand parameters $\boldsymbol{\theta}$. We show that if Ω and $\boldsymbol{\theta}$ remain in a compact neighborhood in which the stage game admits a unique solution, then equilibrium prices denoted by $p^*(\Omega, \boldsymbol{\theta})$ are continuous in Ω and θ. Consequently, a small change in the opportunity costs does not change prices substantially. In the dynamic game, as long as no sales occur, prices do not jump over time provided Ω and $\boldsymbol{\theta}$ stay in the compact neighborhood. This property turns out useful for generalizing Lemma 1 and simulating equilibrium price paths.

Lemma 3. *Let Assumptions 1 and 2 hold. If the equilibrium of the stage game is unique for a compact set of* $(\Omega, \boldsymbol{\theta}) \in \mathcal{O}$*, then there exists an equilibrium price vector* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *for any* $(\Omega, \boldsymbol{\theta})$ *such that* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *is continuous in* $(\Omega, \boldsymbol{\theta})$ *on* $\mathcal{O}$.

Given Assumption 2, Assumption 3-ii) is satisfied for any matrix of scarcity effects Ω in a neighborhood $\mathcal{O}$ that contains the zero matrix $\Omega = \mathbf{0}$ by continuity. However, Assumption 3-ii) can fail for non-zero values of scarcity effects. In such cases, we can get multiplicities of equilibria that can potentially result in price jumps that are not caused by a change in inventory in the dynamic game. The following discussion illustrates this point.

Lemma 3 can fail if Assumption 3 is violated. To see this, consider logit demand such that $\delta_1 = \delta_2 = 0$, and $\rho = 1$. In this case, Assumption 3 is equivalent to

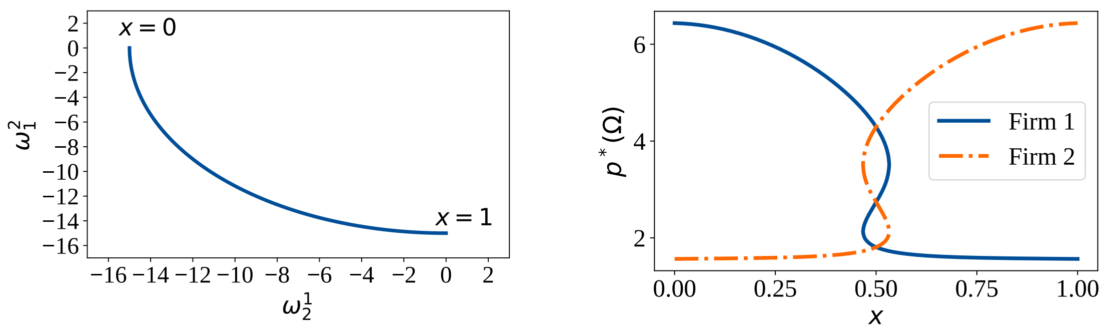
$$\Big(s_1(\mathbf{p}) + \alpha\omega_2^1 s_0(\mathbf{p})\Big)\Big(s_2(\mathbf{p}) + \alpha\omega_1^2 s_0(\mathbf{p})\Big) \neq 1 + \frac{1 - s_1(\mathbf{p}) - s_2(\mathbf{p})}{s_1(\mathbf{p})s_2(\mathbf{p})}.$$

Note that this condition does not depend on the firms' own-product scarcity effects ω_1^1 and ω_2^2. Therefore, we set own-product scarcity effects equal to zero and parameterize competitor scarcity effects using a continuous function. We plot the parameterization of (ω_1^2, ω_2^1) in Figure 1-(a). We plot the corresponding equilibrium prices for both firms in

1-(b). The figure shows that multiplicity of equilibria can occur and there are jumps in prices—even when scarcity effects change continuously.

Figure 1: Multiplicities in stage-game equilibria

(a) Parametrization of $(\omega_2^1(x), \omega_1^2(x))$ (b) Multiplicity in equilibrium prices



Note: In these graphics we parameterize (ω_1^2, ω_2^1) with a curve $(\omega_2^1(x), \omega_1^2(x)) = \left(-15\cos\left(\frac{\pi}{2}x\right), -15\sin\left(\frac{\pi}{2}x\right)\right), x \in [0,1]$, where we set $(\omega_1^1, \omega_2^2) = (0,0)$, and assume logit demand with $\delta = (1,1)$, $\alpha_t = 1$ and scaling factor $\rho = 1$. Panel (a) depicts the parameterized curve and panel (b) equilibrium prices of both firms given (ω_1^2, ω_2^1) at varying values of x.

3.1.3 Characterization of Continuous-time Limit

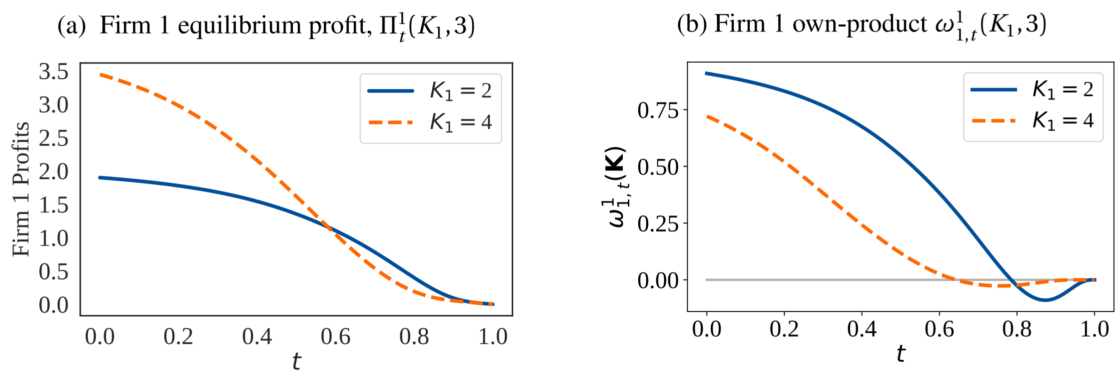
Using Lemma 3 and Lemma 5 in the appendix, we can generalize Lemma 1 to a duopoly as long as the time horizon is not too long. We state the result formally below. The equilibrium characterization is useful because it allows us to simulate equilibrium outcomes in our empirical analysis for high-dimensional games.

Proposition 2 (Continuous-time Limit)**.** *Let Assumptions 1, 2, and 3 hold for* $\Omega = \mathbf{0}$*. For every* $\mathbf{K}$*, there exists a* $T_0(\mathbf{K}) > 0$*, non-increasing in* $\mathbf{K}$*, so that for any* $T \leq T_0(\mathbf{K})$ *there exists a unique equilibrium of the dynamic pricing game for sufficiently small* Δ*. The corresponding value function* $\Pi_{f,t}(\mathbf{K};\Delta)$ *converges to a limit* $\Pi_{f,t}(\mathbf{K})$ *as* $\Delta \to 0$ *that solves the differential equation*

$$\dot{\Pi}_{f,t}(\mathbf{K}) = -\lambda_t \quad \Big(s_{f,t}(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \left(p_f^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t) - (\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_j))\right) \\ - s_{f',t}(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \left(\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_{f'})\right)\Big),$$

where $f' \neq f$, with boundary conditions (i) $\Pi_{f,T}(\mathbf{K}) = 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K}) = 0$ if $K_f = 0$, (iii) $\Pi_{f,t}(\mathbf{K}) = -\infty$ if $K_f < 0$, and (iv) $\Pi_{f,t}(\mathbf{K} - \mathbf{e}_{f'}) = \Pi_{f,t}(\mathbf{K})$ if $K_{f'} = 0$, $K_f \geq 0$.

Figure 2: Simulated profits and own-product scarcity effects when $K_2 = 3$ and K_1 varies



Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$. Panel (a) shows firm 1's profits over time, $t \in [0,1]$, for $\mathbf{K} = (2,3)$ and $\mathbf{K} = (4,3)$. Panel (b) shows firm 2's profits over time, $t \in [0,1]$, for the same states.

The characterization also allows us to illustrate that the general insights from the single-firm setting (Proposition 1) do not hold in an oligopoly. In Figure 2, we consider a simulation using logit demand. We fix the capacity of firm 2 to be $K_2 = 3$ and vary the level of firm 1 capacity K_1 (either 2 or 4). In panel (a), we plot firm 1 profits over time for given capacities. The figure shows that firm 1 expects higher profits with $K_1 = 4$ than with $K_1 = 2$ far from the deadline, however, the firm also expects higher profits with $K_1 = 2$ versus $K_1 = 4$ close to the deadline. That is, the value function is non-monotonic in own capacity. In panel (b), we plot the own-product scarcity effect of firm 1. Contrary to the single-firm case, we see that the own-product scarcity effect is also not monotonic in own capacity. In addition, note that the own-product scarcity effect is negative close to the deadline but positive later on. In Figure 11 in Appendix D, we present additional figures that highlight that the sign of all scarcity effects can be positive or negative.

3.2 Additional Theoretical Results on Dynamic Price Competition

3.2.1 Prices as Strategic Substitutes vs Strategic Complements

In a static Bertrand game with imperfect substitutes, prices are strategic complements for commonly used demand specifications, including logit and nested logit demand systems. Hence, competition unambiguously lowers prices. Due to the presence of competitor scarcity effects, our model results in pricing games that may be strategic substitutes or strategic complements, even for a simple demand systems.

In order to understand the strategic incentives when a competitor changes its price, recall that the first derivative with respect to p_f of firm f's payoff function is given by $p_f \mapsto \frac{\partial s_f}{\partial p_f}\big(p_f - g_f(\mathbf{p})\big)$. By Assumption 1-ii), the first-order condition is satisfied if and only if $p_f = g_f(\mathbf{p})$. Furthermore, by Assumptions 1-i) and 3-i), there is a unique interior maximum of firm f's payoff function for any competitor price $p_{f'}$ and $g_f(\mathbf{p})$ is strictly decreasing.

How does firm f's best-response change if the competitor raises its price? Firm f's best response increases, i.e., the competitor's price is a strategic complement, if $\frac{\partial g_f}{\partial p_{f'}} > 0$ and it decreases, i.e., the competitor's price is a strategic substitute, if $\frac{\partial g_f}{\partial p_{f'}} < 0$. Typically, the literature assumes monotonicity of the own-price elasticity in the competitor's price, which is, for example, guaranteed for log-concave demands. In this case, prices are strategic complements. However, in our setting, the strategic forces are less straightforward due to the competitor scarcity effect.[11] Given the payoffs in the stage game, we have

$$\frac{\partial}{\partial p_{f'}} g_f(\mathbf{p}) = \frac{\partial}{\partial p_{f'}} \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} \omega^f_{f'} - \frac{\partial}{\partial p_{f'}} \Bigg(\underbrace{ s_f(\mathbf{p}) \left(\frac{\partial s_f(\mathbf{p})}{\partial p_f} \right)^{-1} }_{\substack{\text{inverse quasi} \\ \text{own-price elasticity}}} \Bigg). \tag{7}$$

To gain some concrete intuition, it is useful to to consider the simple logit specification

[11] As has been noted for example by Vives (2018); Nocke and Schutz (2018), in general, static oligopoly games in multi-product environments are, however, not games of strategic complements.

introduced in Section 2.1. Then, we have that

$$\frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} = -\frac{s_{f'}(\mathbf{p})}{1 - s_f(\mathbf{p})} = -\frac{\exp\{\delta_{f'} - \alpha p_{f'}\}}{1 + \exp\{\delta_{f'} - \alpha p_{f'}\}},$$

which corresponds to the negative of the competitor's demand if firm f's product is excluded from the market. By Assumption 1, this expression is increasing in $p_{f'}$.[12] Plugging this expression into Equation (7), it follows that if the competitor scarcity effect $\omega^f_{f'}$ is positive, an increase in competitor price increases a firm's cost of selling a product. This puts upward pressure on the own price p_f. In contrast, for negative $\omega^f_{f'}$, the cost is decreasing in the competitor's price—firm f benefits if the competitor sells a unit. In an extreme case, when $\omega^f_{f'}$ is very negative, it might be that $\frac{\partial g_f}{\partial p_{f'}} < 0$. As a result, the competitor's price can became a strategic substitute to the firm's own price. See Figure 12 in Appendix D for an example using logit demand where adjusting $\omega^f_{f'}$ changes the stage game from being one of strategic complements to strategic substitutes.

3.2.2 The Influence of Remaining Capacities on Prices

Next, we link remaining capacity to incentives to soften price competition. We focus on a demand system that is constant over time, i.e., $\lambda_t = \lambda$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}$, to single out the effects of remaining capacities. However, similar forces occur with time-dependent parameters, as we describe in our empirical analysis.

Starting at the deadline, note that equilibrium prices, $\mathbf{p}^*_T$, are equal to the stage game equilibrium prices where all scarcity effects are zero. As we move away from the deadline, capacity can influence pricing dynamics. We establish that the order of change of prices towards the deadline is determined by the product with the minimum capacity in the market. The order of change is reduced by one only if a unit of the product with the minimum capacity is sold. The proposition is formally stated below.

[12] A similar expression can be derived for the more flexible nested logit demand specification as shown in Appendix C.

Proposition 3. *Let $\lambda_t \equiv \lambda$, $\alpha_t \equiv \alpha$. Then, for* $\mathbf{K}$ *with* $\underline{K} := \min_f K_f$*, the following holds:*

$$p_{f,t}(\mathbf{K}) = p^*_{f,T} + \mathcal{O}(|T-t|^{\underline{K}}),\ t \to T \ \text{for } f = 1,2,$$

i.e., price changes close to the deadline are at most of order $\underline{K}$. *If* $\lim_{t\to 0} \frac{\partial^{\underline{K}}}{(\partial t)^{\underline{K}}} \Pi_{h,t}(\mathbf{K}-\mathbf{e}_h) \neq 0$ *for all f with $K_h = \underline{K}$, then*[13]

$$p_{f,t}(\mathbf{K}) = p^*_{f,T} + \Theta(|T-t|^{\underline{K}}),\ t \to T \ \text{for } f = 1,2,$$

i.e., price changes are exactly of order $\underline{K}$.

The proposition establishes that prices change the most after the sale of the firm with the lower capacity as the prospect of significant future price changes are manifested in the scarcity effects of the product with minimum capacity. If capacities are asymmetrically distributed, the competitor scarcity effect of the firm with more capacity is negative because a sale of the competitor increases prices significantly. In contrast, the competitor scarcity effect of the firm with less capacity is small because prices remain nearly unchanged even if the competitor sells. Thus, only the firm with more capacity has an incentive to set a higher price. The firm with less capacity has an incentive to set a lower price.

In contrast, if firms have the same capacity, then any sale leads to a price jump, regardless of which firm sells. That is, firms would like to sell a unit in order to soften price competition, but with symmetric capacities, both competitor scarcity effects are negative and own-product scarcity effects are small (and possibly negative). Although firms would like to set high prices competitor scarcity effects are negative), there is a countervailing force: high prices implies the probability of any sale is low, and hence, firms would be unable to soften future price competition unless one firm sells. Therefore, firms actually engage in fierce competition—they set very low prices, possibly even lower than $\mathbf{p}^*_T$, in order to leave the competitive state quickly.

[13] Recall that $f(t) = \mathcal{O}(g(t))$ as $t \to T$ if $\exists \delta, C_1 > 0$ so that for all t with $0 < |T-t| < \delta$, $|f(t)| \leq C_1 g(t)$. $f(t) = \Theta(g(t))$ if additionally $\exists C_2 > 0$ so that $C_2 g(t) \leq |f(t)|$.

We illustrate these price competition effects in Figure 14 in Appendix D. We consider firms with $\mathbf{K}$ =(5,4); (4,4); and (3,4) capacities. Note that (4,4) prices are the lowest, and the firm with the lowest capacity XX[14]

Empirically, this means that we expect firms to benefit from dynamic pricing whenever remaining capacities are distributed unequally across firms, as equally distributed capacities result in intense price competition.

3.2.3 Independence of Irrelevant Alternatives and Markup Formula

Finally, we show that for demand specifications that satisfy the commonly used assumption of "Independence of Irrelevant Alternatives (IIA)," the stage game admits an equilibrium for any scarcity matrix Ω. Moreover, the game satisfies a markup formula.

Assumption 4 (Independence of Irrelevant Alternatives (IIA))**.** *Suppose the following holds,*

$$\frac{\partial}{\partial p_1}\frac{s_2(\mathbf{p})}{s_0(\mathbf{p})} = \frac{\partial}{\partial p_2}\frac{s_1(\mathbf{p})}{s_0(\mathbf{p})} = 0.$$

Given Assumptions 1, 2 and 4, we establish the following proposition:[15]

Proposition 4 (Mark-up formula under IIA)**.** *Let Assumptions 1, 2 and 4 hold. Then, there exists an equilibrium $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ of the above stage game for any scarcity matrix Ω. Further, there exist functions $d_f(p_{f'}; \boldsymbol{\omega}, \alpha)$, $f \neq f'$ so that $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ is the unique price vector such that*

$$p_f^*(\Omega, \boldsymbol{\theta}) \in \arg\max_{p_f} s_1(p_f, p_{f'}^*(\Omega, \boldsymbol{\theta}))(p_f - c_f(p_{f'}; \Omega, \boldsymbol{\theta})) + d_f(p_{f'}; \Omega, \boldsymbol{\theta})$$

for $f \in \{1,2\}$, $f' \neq f$, where $c_f(p_{f'}; \omega, \boldsymbol{\theta}) := \omega_f^f + \tilde{s}_{f'}(p_{f'})\omega_{f'}^f$, and $\tilde{s}_{f'}(p_{f'}) := \frac{s_{f'}(\mathbf{p})}{s_0(\mathbf{p})}$ is the demand of firms f' conditional on firm f not selling.

[14]The relationship between prices and competing firms' inventories has been explored in other contexts, e.g., see Israeli et al. (2022) on car dealership pricing.

[15]The general result in Appendix A additionally shows that with multiple products for each firm, the game can be transformed to a game in which each product is managed by its own firm given transformed payoff functions.

Proposition 4 implies that equilibrium prices $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ satisfy a markup formula

$$\frac{p_f^*(\Omega, \boldsymbol{\theta}) - c_f(p_{f'}; \Omega, \boldsymbol{\theta})}{p_f} = -\frac{1}{\epsilon_f(\mathbf{p})}, \tag{8}$$

where $\epsilon_f(\mathbf{p}) = \frac{\partial s_f(\mathbf{p})}{\partial p_f} \frac{p_f}{s_f(\mathbf{p})}$ is the elasticity of demand. Equation (8) shows that price dynamics are governed by the evolution of the net opportunity cost c_f and the change in demand elasticity. The net opportunity cost $c_f(p_{f'}; \Omega, \boldsymbol{\theta})$ is a function of both the own-product scarcity effect and the competitor scarcity effect, weighted by the relative market share of the competitor relative to the outside option. Thus, if many consumers pick the outside option, or if the competitor is small, a firm's decision is less affected by the competitor. If the competitor is large, the competitor scarcity effect has a larger weight, putting more upwards pressure on own prices.

4 Data and Descriptive Evidence

4.1 Data Description

Our empirical insights are derived from data provided to us through a research partnership with a large U.S. airline.[16] The core data set contains booking and pricing information covering competing airlines and was assembled by third parties that collect and combine contributed data. The data have strong parallels with other contributed data sets, such as the the Nielsen scanner data used to study retailing, in that we observe prices and quantities for competing firms.

The bookings data track flight-level sales counts over time. We use the tuple j, t, d to denote an airline-flight number, day before departure, departure date combination. The frequency of the data is daily. We observe separate booking counts for passengers flying between an origin-destination pair (OD) and consumers making connections. We call these consumers *local* and *flow* passengers, respectively. Our structural analysis focuses on lo-

[16]The airline has elected to remain anonymous.

cal, nonstop traffic. We do not model the potential for consumers to connect while flying between an origin-destination pair.

We observe bookings for consumers who purchased directly with the airline and on other booking channels, e.g., online travel agencies. We label these bookings *direct* and *indirect*, respectively. Because we observe all booking counts, we can construct the load factor for each flight over time. We do not know the exact itinerary involved for each booking, e.g., a round-trip versus a one-way booking. Therefore, we assume that the price paid for each nonstop booking corresponds to the lowest available nonstop, one-way fare for that flight.

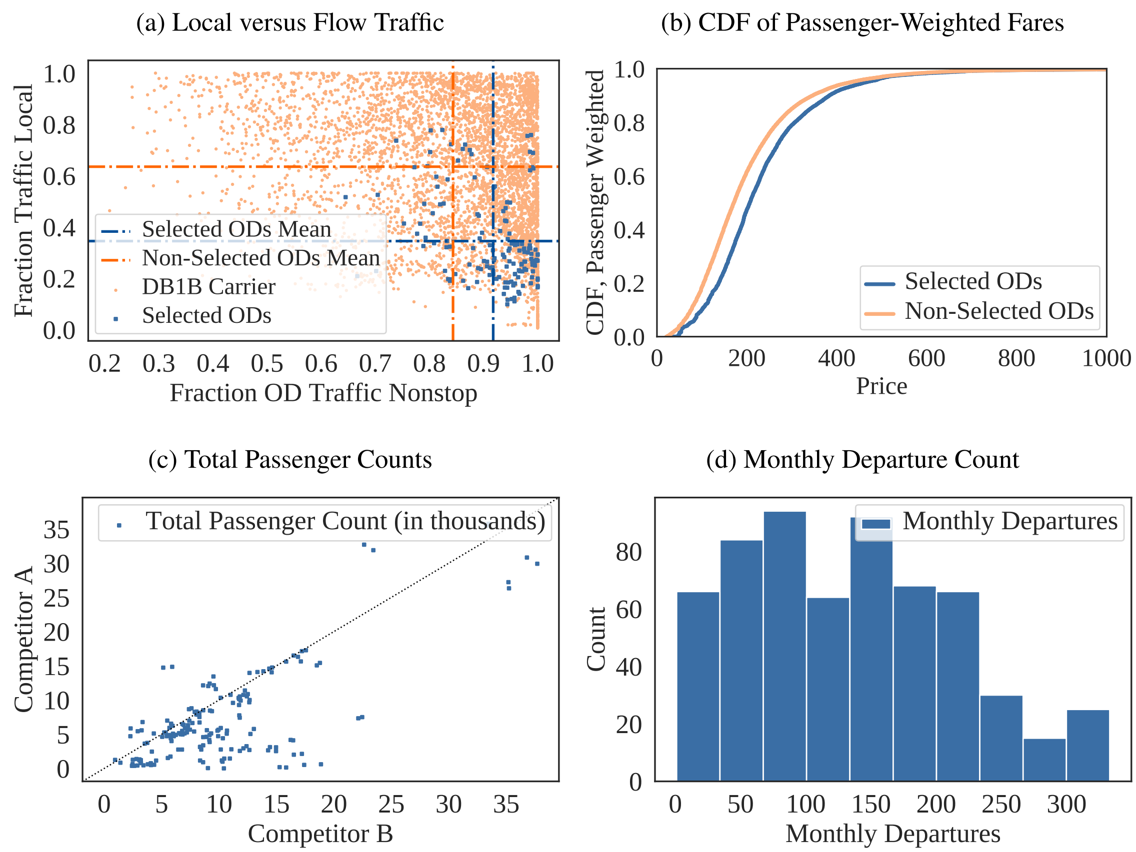
Our pricing data come from a separate third-party data provider that gathers and disseminates fare information for the airline industry. The data frequency matches the booking information, i.e., we observe daily prices at the flight level. We observe fares even when there are no bookings. Several prices are tracked, including tickets of different qualities (cabins, fully refundable, etc.). We concentrate our analysis on the lowest available economy class ticket because travelers overwhelmingly purchase the lowest fare offered (Hortaçsu et al., 2021b). We do not model consumers choosing between cabins (economy vs. first class) nor the pricing decision for different versions of tickets.

In order to gauge market sizes, we use clickstream search data provided to us by the air carrier. See Hortaçsu et al. (2021a) and Hortaçsu et al. (2021b) for more details. Observed searches understate true arrivals because some consumers may search and purchase through online travel agencies or directly with competitors. We extrapolate total arrivals by scaling up observed searches using hyperparameters that we describe below.

4.2 Route Selection

Our analysis concentrates on nonstop flight competition. We limit ourselves to routes where nonstop service is provided by exactly two airlines—by our data provider and one competitor. Our data contain more than one competitor airline, however, we will always refer to the competing airline as “the competitor.” We eliminate routes where the third-party data is incomplete, e.g., where a carrier provides direct bookings to the data provider but indirect bookings are missing. In addition to these criteria, we select routes in which most OD traffic is traveling nonstop. This selection criteria allows us to avoid the additional complexity of modeling connecting traffic.

Figure 3: Summary Analysis from the DB1B Data



Note: Panel (a) records the percentage of flow (connecting) vs local traffic and the percentage of non-stop traffic in the DB1B data. Panel (b) plots the cdf of prices for selected routes and all dual-carrier markets. Panel (c) reports total passenger counts for both competitors. Panel (d) reports the number of aggregate monthly departures for the routes in our sample.

In Figure 3 we provide summary analysis of the 58 routes in our data using the publicly available DB1B data. These data contain 10% of bookings in the U.S. but lack information on the booking and departure date. In panel (a), we show the percentage of total traffic that is local versus the percentage of local traffic flying nonstop for our data compared to all dual-carrier nonstop markets in the U.S. The selected markets primarily contain local traffic that are traveling nonstop. In panel (b) we show that the distribution of fares in our markets is similar to the universe of dual-carrier markets.

In panels (c) we use the DB1B data to compute the quarterly passenger counts of the competing airlines in our data set. The panel shows the total passenger count for "Competitor A" and "Competitor B," which we use to denote our air carrier and the nonstop competitor, respectively. Each dot represents an OD-quarter. The panel shows the diversity of routes in our sample. There is considerable variation in the total size of the market (distance from the origin) as well as the relative size of the airlines for each OD. There is also variation in the passenger count of nonstop traffic within an OD across carriers.

Finally, in panel (d) we use the publicly available T100 segment data to plot the total number of monthly departures for the routes in our sample. Over half of our sample contains routes in which there are less than five daily frequencies between the origin and destination. Several routes feature twice daily service (one flight per airline). At the other extreme, one route in our data contains nearly 10 daily frequencies.

4.3 Descriptive Evidence

Table 1: Summary statistics

Data Series	Variable	Mean	Std. Dev.	Median	5th pctile	95th pctile
Fares						
	One-Way Fare ($)	233.7	111.4	218.6	92.1	390.7
	Num. Fare Changes	6.4	2.4	6.0	3.0	11.0
Bookings						
	Booking Rate-local	0.2	0.6	0.0	0.0	1.0
	Booking Rate-all	0.5	1.2	0.0	0.0	3.0
	Ending LF (%)	72.1	19.8	76.0	32.9	98.0

Note: One-Way fare is for the lowest economy class ticket available for purchase. Number of fare changes records the number of price adjustments observed for each flight. Booking rate-local excludes flow traffic. Booking rate-all includes both local and flow traffic. Ending load factor (LF) reports the percentage of seats occupied at departure time.

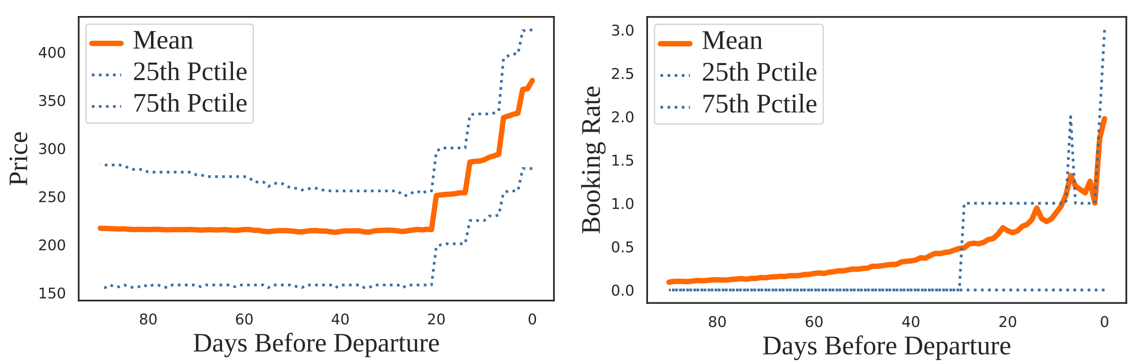
We provide a summary of the main data in Table 1. All flights departed in the first nine months of 2019. Average fares across airlines in our sample are $233. On average, each flight experiences about six price adjustments within 90 days. In terms of bookings, the average daily booking rate is less than one. Roughly 40% of observed bookings are for local traffic, the remaining are flow bookings. At the departure time, average load factors are 72%, which is lower than the industry average of about 80% for this time period. We do observe sellouts for all competitors in the data.

In Figure 4 we plot average fares and booking rates by day before departure. The left panel (prices) shows that average fares are fairly flat between 90 and 21 days before departure. The top end of the distribution is decreasing in this time window. There are noticeable "steps" in the last 21 days before departure which highlights the use of advance purchase (AP) discounts in the industry. In the routes examined, we observe AP requirements at 21, 14, 7, and 3 days before departure. In the right panel (bookings) we highlight that bookings increase as the departure date approaches. This coincides with increasing prices and suggests that demand becomes more inelastic over time. The booking rate is greater than one per flight over the last month before departure.

Figure 4: Prices and Bookings by Day Before Departure

(a) Prices over Time

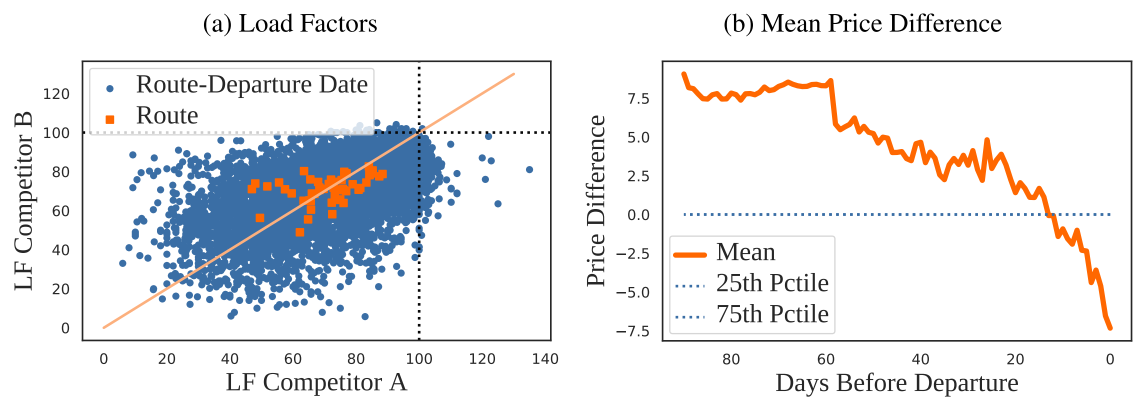
(b) Bookings per Route-Departure Date over Time



Note: Panel (a) shows the average and interquartile range of flight prices over time. Panel (b) shows the average and interquartile range of flight booking rates over time. Greater than 30 days before departure, the 25th and 75th percentiles coincide.

In Figure 5 we focus on outcomes across competitors. The left panel provides a scatter plot of ending load factor at the route-departure date level for the entire data sample. The orange squares present route-level load factors. Note there exists a large mass of points both above and below the 45-degree line—one competitor does not consistently sell a larger fraction of capacity than the other carrier for all routes. We do observe some flights with substantial overselling. In our analysis, we restrict firms to selling at most their capacity. In the right panel we plot the average fare difference across competitors over time when exactly two flights are offered. Note that fares tend to be similar across competitors—the average difference is less than \$10. However, the gradient of the prices differs. One competitor has relatively higher prices well in advance of departure and relatively lower prices close to departure. Note that for over 50% of the data, prices across firms are equal, that is, there is substantial price matching.

Figure 5: Load Factor and Price Differences across Carriers



Note: Panel (a) shows the average load factor (across all flights) at the route-departure date level for both competitors in blue. The orange squares report average route-level load factors. The diagonal line is the 45-degree line. Panel (b) shows the average and the 25th and 75 percentiles of the difference in prices for markets in which exactly two flights across firms are offered (one flight per airline).

5 Demand Model and Estimates

5.1 Empirical Specification

We model nonstop air travel demand using a nested logit demand model. Our model differs from recent empirical work on airlines that use a mixed-logit model to model "business" and "leisure" travelers (Lazarev, 2013; Williams, 2022; Aryal et al., 2021; Hortaçsu et al., 2021b). We use a flexible nested-logit model with time-varying as it better maps to our theoretical model and results in unique equilibrium price paths.[17]

Define a market as an origin-destination (r), departure date (d), and day before departure (t) combination. Each flight j, leaving on date d, is modeled across $t \in \{0, ..., T\}$. The first period of sale is $t = 0$, and the flight departs at T. We use a 90-day time horizon. With daily data, we model demand at the daily level. Arriving consumers choose flights from the choice set $J_{t,d,r}$ that maximize their individual utilities, or select the outside option, $j = 0$. There are two nests. The outside good belongs to its own nest, and all inside goods belong to the second nest.

We specify consumer arrivals to be

$$\lambda_{t,d,r} = \exp\left(\tau_r^{\text{OD}} + \tau_d^{\text{DD}} + \tau_{t,d}^{\text{SD}} + f\,(\text{DFD})_t\right),$$

where τ denote fixed effects for the route, departure date, and search date; $f(\cdot)$ is a polynomial series of degree three. We scale up these estimated arrival rates using hyperparameters to account for unobserved searches.

Conditional on arrival, we specify consumer utilities as

$$u_{i,j,t,d,r} = \mathbf{x}_{j,t,d,r}\boldsymbol{\beta} - \alpha_t p_{j,t,d,r} + \zeta_{i,J} + (1-\sigma)\varepsilon_{i,j,t,d,r},$$

where $\zeta_{i,J} + (1-\sigma)\varepsilon_{i,j,t,d,r}$ follows a type-1 extreme value distribution, and $\zeta_{i,J}$ is an idiosyncratic preference for the inside goods. The parameter $\sigma \in [0,1]$ denotes correlation in preferences within the nests. We allow the price sensitivity parameter to vary over time (α_t) using three-day intervals of time; hence, we estimate 30 price sensitivity parameters. We include a number of covariates in $\mathbf{x}$ where preferences are assumed to not vary across t: departure week of the year, departure day of the week, route, carrier, and departure time fixed effects.

Each arriving consumer solves their utility maximization problem such that consumer

[17] The mass-point random coefficients models yields multiple equilibria in our setting.

i chooses flight j if, and only if,

$$u_{i,j,t,d,r} \geq u_{i,j',d,t,r}, \forall j' \in \mathscr{J}_{t,d,r} \cup \{0\}.$$

Temporarily dropping the t, d, r subscripts, we define

$$D_J := \sum_{j \in J} \exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta} - \alpha_t p_j}{1-\sigma}\right),$$

so that the probability that a consumer purchases j within the set of inside goods is equal to

$$s_{j|J} = \frac{\exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta}_t - \alpha_t p_j}{1-\sigma}\right)}{D_J}.$$

It follows that the probability that a consumer purchases any inside good product is equal to

$$s_J = \frac{D_J^{1-\sigma}}{1 + D_J^{1-\sigma}}.$$

We define overall product shares to be equal to $s_j = s_{j|J} \cdot s_J$, which are implicitly at the market level (t, d, r).

Our assumptions imply that demand is distributed Poisson with a product purchase rate equal to $\min\left\{\lambda_{t,d,r} \cdot s_{j,t,d,r}, C_{j,t,d,r}\right\}$. Note as the length of a period decreases, at most one seat will be sold in any period.

5.2 Demand Estimates

We estimate the model in two steps. In the first step, we estimate the arrival process parameters using Poisson regressions. We then estimate preferences of the Poisson demand model using maximum likelihood. We estimate standard errors using bootstrap.

We follow Hortaçsu et al. (2021b) in constructing arrivals using clickstream data for one airline. These data track all "clicks" or interactions on the firm's websites. We first sum the number of searches corresponding to each market (r, d, t) and then we scale up estimated arrival rates to account for unobserved searches. This follows from a property of the Poisson distribution and the assumption that consumers who search/purchase through alternative platforms (travel agents, other carriers' websites) have the same underlying preferences. We first calculate the fraction of direct bookings by day before departure and then scale up the estimated arrival rates using these these fractions. This adjusts arrivals for a single carrier. In our preferred specification, we then double these arrival rates to account for competitor indirect and direct searches, both of which are unobserved to us. We conduct robustness to this hyperparameter in Appendix D.

We summarize the demand estimates in Table 2. We estimate the nesting parameter to be 0.5 so that there is substantial substitution within inside goods. The price sensitivity parameters vary by nearly a factor of ten over time. We present a time series plot of α_t in Figure 6. Almost all of our controls are significant, with day of the week and week of the year having the most influence on market shares. The competitor FE is significantly less important in driving variation in shares. We estimate the average own-price elasticity to be -1.4.

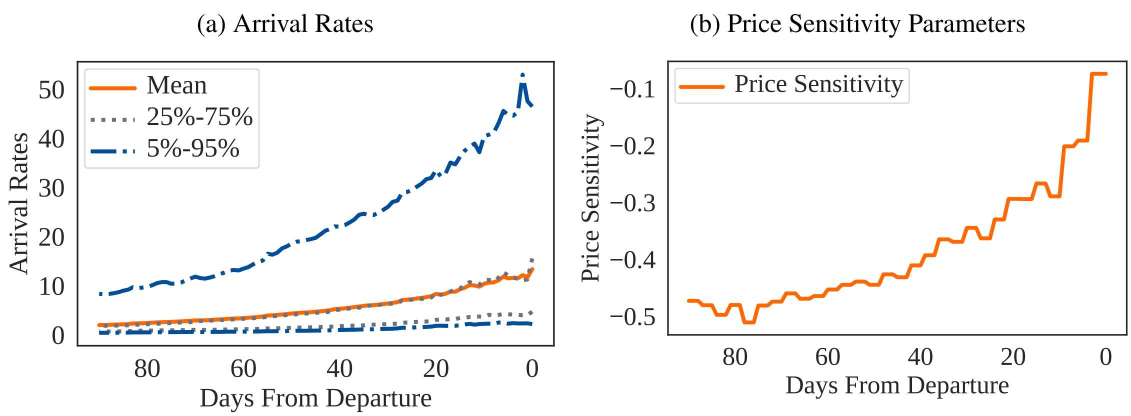
Table 2: Demand Estimates Summary Table

Variable	Symbol	Estimate	Std. Error.	Range	% Sig.
Nesting Parameter	σ	0.498	0.010	–	–
Price Sens.	α	–	–	[-0.511 ,-0.074]	100.0
Competitor FE	–	0.071	0.003	–	–
Day of Week FE	–	–	–	[-1.637 ,-0.961]	100.0
Departure Time FE	–	–	–	[-0.462 ,-0.050]	100.0
Route FE	–	–	–	[-0.177 ,0.226]	94.4
Week FE	–	–	–	[-0.953 ,0.699]	86.0
Sample Size	N		2,814,686		
Avg Elasticity	e^D		-1.438		

Note: Demand estimates for the 58 routes in our sample.

In Figure 6-(a), we plot average adjusted arrival rates as well as parts of the distribution (5%, 25%, 75%, 95%) across markets. We estimate just a few arrivals per market 90 days before departure that then increases to over 10 passengers per day close to departure. Recall that the average booking rate across flights is less than 2.0 (see Figure 4) so that market shares are low. An increase in interest in travel is a general findings across all of the routes in our sample. Note that while the 75th percentile closely followed the mean, the top part of the distribution is substantially higher, which corresponds to the routes in the upper-right of Figure 3-(d).

Figure 6: Arrival Rates and the Price Sensitivity Parameters



Note: Panel (a) shows fixed values, adjusted for unobserved searches, of arrival rates over time. The mean is the average arrival rate across all markets. The percentiles are also over markets. Panel (b) shows our estimates of the price sensitivity parameters in 3-day groupings.

6 Counterfactual Analysis

With our demand estimates, we quantify the welfare effects of dynamic price competition using three sets of counterfactuals—the benchmark, lagged, and deterministic models presented in our theoretical analysis.

Although the benchmark model holds for an arbitrary number of firms and products, computing equilibria of the game is difficult. We adjust our empirical estimates in a number of ways for computational reasons:

i) We consider only two products. Instead of investigating pricing in routes where we observe a single flight operating by each firm, we adjust the choice set, utilities, and capacities for all routes.

ii) We take the mean utilities across observed flights for each departure date and an input.

iii) We take the maximum observed capacity for each route-carrier-departure date. Although it may be natural to sum the capacities when restricting the choice set, we have found that large capacities presents a significant computational burden.

iv) We use the observed arrival process for each route-departure date. We do not adjust the estimated arrival processes as the inside good shares tend to be small. That is, because most consumers choose the outside good, we do not scale down arrival rates to account for smaller choice sets.

v) Finally, we handle flow (connecting traffic) bookings two ways. In our reported counterfactuals here, we model these bookings via Poisson processes that the firm does not internalize when pricing local demand. In the appendix we report counterfactuals where we subtract off all connecting bookings at the start of the game. This affects market outcomes because it reduces uncertainty for firms.

Benchmark Model

We approximate the continuous time model to solve for equilibrium prices for every departure date. We consider hourly decisions over 90 days. Both firms start with initial capacities K_f and $K_{f'}$. We solve via backward induction, which we outline here. In the last pricing period, $t = T$, both $\Pi_T(\mathbf{K}) = 0$ and $\Omega_T(\mathbf{K}) = \mathbf{0}$. Therefore, both firms solve static revenue maximize problems. We set the first-order conditions corresponding to the best response functions equal to zero and solve for the fixed point. We denote the fixed-point price vector by $\mathbf{p}_T = \mathbf{p}^*(\Omega_T, \alpha_T)$ where $\Omega_T = 0$. Let us denote the stage-game payoff of firm f in period t

given price vector $\mathbf{p}$ and Ω by $\pi_{f,t}(\mathbf{p},\Omega)$. Then, using the differential equation, we can write $\dot{\Pi}_{f,T}(\mathbf{K}) = -\lambda_T \pi_{f,T}(p_T(\mathbf{K}),\mathbf{0})$, which allows us to calculate $\Pi_{f,T-\Delta}(\mathbf{K}) = \Pi_{f,T}(\mathbf{K}) - \Delta\cdot\dot{\Pi}_{f,T}(\mathbf{K})$ and $\omega^f_{f',T-\Delta}(\mathbf{K}) = \Pi_{f,T-\Delta}(\mathbf{K}) - \Pi_{f,T-\Delta}(\mathbf{K}-\mathbf{e}_{f'})$. Given the updated own and scarcity effect parameters we again solve for equilibrium prices, $\mathbf{p}_{T-\Delta} = \mathbf{p}^*_{T-\Delta}(\Omega_{T-\Delta},\alpha_{T-\Delta})$.[18] We continue the induction backwards in time to $t = 0$.

Due to the large number of state variables, we store Ω_t and $\mathbf{p}_t$ every 24 hours (at the start of a day) in order to use them in counterfactual simulations. We then appeal to modeling demand via multinomial distributions after drawing arrivals from a Poisson distributions in lieu of studying each consumer's individual choice after drawing arrivals from a Bernoulli distributions as in the theoretical model.

6.0.1 Pricing with Heuristics

We compare the benchmark model to two pricing heuristics where firms do not internalize the scarcity of their competitor and firms also do not explicitly account for the fact that their competitor is a strategic agent solving a dynamic pricing problem. In both heuristics, we consider discrete prices as they are used in actual airline pricing practices. Applied theory work, including Asker et al. (2021), also consider discrete prices. The pricing menu (set of discrete prices for all time periods) is taken as given.

We label the heuristics "Lagged Model" and "Deterministic Model," respectively. In the lagged model, each firm, having observed its competitor's last period price, assumes this price will also be charged in the current and all future periods. Each firm then calculates its residual demand curves in all remaining periods and solves a single-firm dynamic programming problem. In the deterministic model, each firm simply assumes its competitor will price at the lowest possible price in all remaining periods. Our simulations show that price levels can be higher or lower than in the benchmark model depending on demand parameters.[19] In the next section we will feed in estimated demand models into the bench-

[18] We use a modified Powell method from MINPACK's hybrid routine to solve the system of first order conditions corresponding to the best-response functions.

[19] Interested readers can find example price paths of the heuristics in Figure 16 in Appendix D.

mark model and heuristic models to empirically estimate the implications of profits, total welfare, consumer surplus, and prices.

Lagged-Price and Deterministic Models

Recall that both the lagged-price and deterministic models use discrete fares. All airlines use discrete fares, and our data allow us to create fare menus for all carrier, route combinations.[20] More specifically, airlines file fares for "buckets." Typically, each carrier fills between seven and fifteen buckets per route. Buckets can change by day before departure, i.e., the fare for a given bucket increases. However, the data suggests that a more consequential change in buckets over time is their availability. Oftentimes, a fare is restricted for a certain time period before departure—an advance purchase discount.

Figure 15-(a) in Appendix D shows an example fare menu for a given carrier-route in the data. Prices vary from less than \$200 to over \$3,000. In Figure 15-(b) and Figure 15-(c), we provide example price paths for the lagged and deterministic models using our empirical estimates.

Implementation

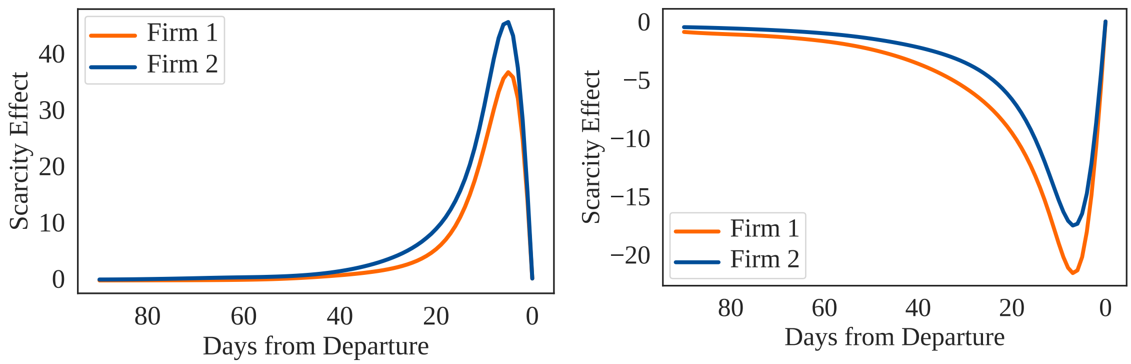
We conduct 10,000 Monte Carlo experiments for every route, departure date combination. We simulate all counterfactuals twice, one where flow traffic is subtracted from initial observed capacity is advance, and one where flow traffic is modeled through Poisson processes, not internalized when pricing local demand. We store prices, arrivals, quantities sold, and calculate consumer surplus and revenues for every market.

6.1 Welfare Effects of Dynamic Price Competition

[20] See Hortaçsu et al. (2021b) for more details.

Figure 7: Benchmark Model Opportunity Costs

(a) Own Omega (b) Competitor Omega



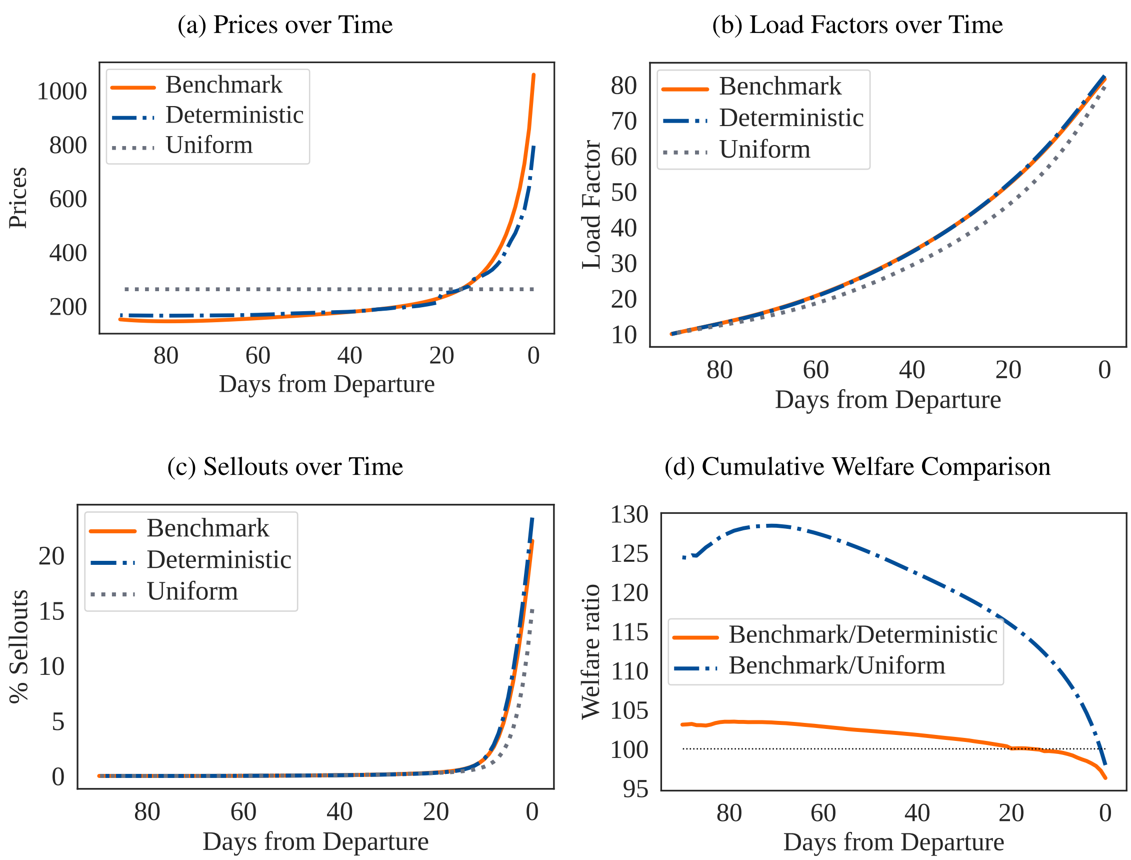
Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Table 3: Counterfactual Results

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Deterministic	98.3	96.8	97.6	108.4	103.9	103.2	101.2	109.9
Lagged	105.2	101.7	102.7	103.9	103.2	99.6	99.9	98.8
Uniform	118.2	85.7	87.4	112.9	102.2	93.6	97.5	72.6

Note:

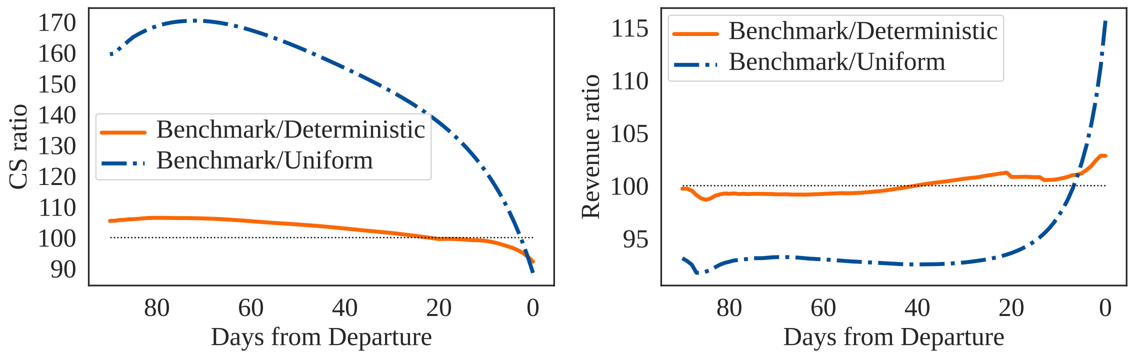
Figure 8: Counterfactual Summary Plots



Note: Panel (a) shows the average price over time for the benchmark, deterministic and uniform models. Panel (b) shows the average load factors over time for the same three models. Panel (c) shows the average sellouts over time for the same three models. Panel (d) shows the ratio of average cumulative welfare for the benchmark model with respect to the deterministic one, and for the benchmark model with respect to the uniform one.

Figure 9: Cumulative Surplus Differences Across Counterfatuals

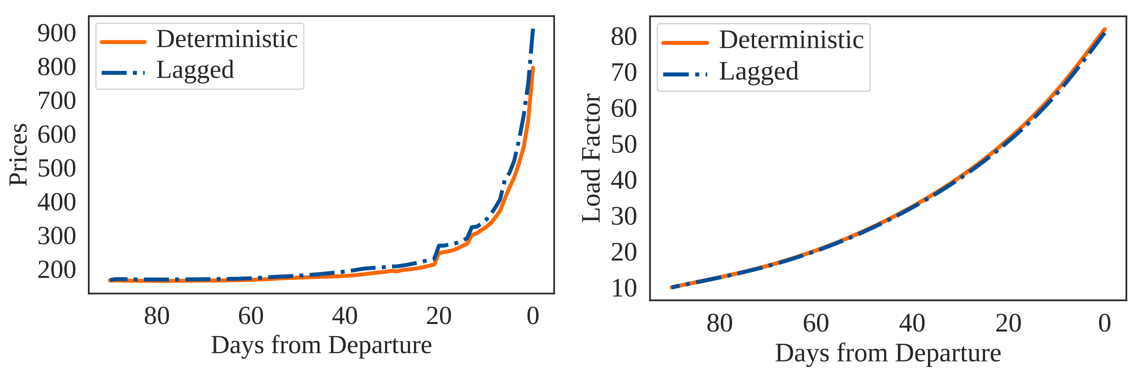
(a) Cumulative CS Difference (b) Cumulative Revenue Difference



Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Figure 10: Heuristic Counterfactuals Prices and Load Factors

(a) Prices over Time (b) Load Factors over Time



Note: Panel (a) shows the average prices over time for the two heuristic models. Panel (b) shows the average load factors over time for the same two models.

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A General Model with Many Firms and Many Products

In Appendix A, we formulate the generalized results stated in Section 3 for the duopoly case. We directly prove those general statements in Appendix B.

A.1 Model Setup

We consider a market with $F \geq 1$ firms and $J \geq F$ products, denoting the set of firms by $\mathscr{F} := \{1, \ldots, F\}$ and the set of products by $\mathscr{J} := \{1, \ldots, J\}$. Each firm f sells products in $\mathscr{J}_f$, where $(\mathscr{J}_f)_{f \in \mathscr{F}}$ is a partition of $\mathscr{J}$; that is, $\mathscr{J} = \bigcup_{f \in \mathscr{F}} \mathscr{J}_f$ and $\mathscr{J}_f \cap \mathscr{J}_{f'} = \emptyset$ for $f \neq f'$. Thus, no product is sold by more than one firm. Each firm f is equipped with an initial inventory of their products $j \in \mathscr{J}_f$, denoted by $K_{j,0} \in \mathbb{N}$. We assume that the demand system for the products in $\mathscr{J}$ is as introduced in Section 2.1, and satisfies Assumptions 1 and 2.

The dynamic pricing game is the canonical generalization of the duopoly game introduced in Section 2.3. In every period t, each firm f simultaneously sets prices $p_{j,t}$ for its products $j \in \mathscr{J}_f$, and then a consumer arrives with probability $\Delta\lambda_t$. If a consumer arrives, she buys product j with probability $s_{j,t}(\mathbf{p}_t)$.

Like for the duopoly, the payoff-relevant state is given by the vector of inventories $\mathbf{K} := (K_j)_{j \in \mathscr{J}}$ and the time t. We study Markov perfect equilibria in which each firm's strategy is measurable with respect to $(\mathbf{K}, t)$. We denote a Markov pricing strategy of firm f by $\mathbf{p}_{f,t}(\mathbf{K}) = (p_{j,t}(\mathbf{K}))_{j \in \mathscr{J}_f}$.

Given equilibrium price vectors $\mathbf{p}^*_t(\mathbf{K}) := (p^*_{j,t}(\mathbf{K}))_{j \in \mathscr{J}}$, firm f's value function satisfies[21]

$$\begin{aligned}\Pi_{f,t}(\mathbf{K};\Delta) = \; & \Delta\lambda_t \Bigg(\underbrace{\sum_{j \in \mathscr{J}_f} s_{j,t}(\mathbf{p}^*_t(\mathbf{K}))\,(p^*_{j,t}(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)) +}_{\text{revenue from own sale}} \\ & \underbrace{\sum_{j' \neq \mathscr{J} \setminus \mathscr{J}_f} s_{j',t}(\mathbf{p}^*_t(\mathbf{K}))\,\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{j'};\Delta)}_{\text{continuation value if } j' \text{ sells}} \Bigg) + \underbrace{\Bigg(1 - \Delta\lambda_t \sum_{j' \in \mathscr{J}} s_{j'}(\mathbf{p}^*_t(\mathbf{K}))\Bigg)}_{\text{probability of no purchase}} \; \Pi_{f,t+\Delta}(\mathbf{K};\Delta),\end{aligned}$$

[21] Formally, equilibrium prices are a function of Δ, which we omit here for readability.

with boundary conditions (i) $\Pi_{f,T+\Delta}(\mathbf{K};\Delta) \equiv 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) \equiv 0$ if $K_j = 0$ for all $j \in \mathcal{J}_f$ and (iii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathcal{J}_f$, (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j;\Delta) = \Pi_{f,t}(\mathbf{K};\Delta)$ if $K_j = 0$ for a $j \notin \mathcal{J}_f$ $K_j \geq 0$ for all $j \in \mathcal{J}_f$.. Then, we denote the *scarcity effect of product* j *on firm* f in state $(\mathbf{K}, t)$ by

$$\omega^f_{j,t}(\mathbf{K}) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta).$$

Then, the stage game is parameterized by the matrix of scarcity effects

$$\Omega_t(\mathbf{K}) = (\omega^f_{j,t}(\mathbf{K}))_{f,j} \in \mathbb{R}^{\mathcal{F}} \times \mathbb{R}^{\mathcal{J}},$$

where firm f's flow payoff $\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta)$ is equal to

$$\Delta\lambda \sum_{j \in \mathcal{J}_f} s_{j,t}(\mathbf{p}^*_t(\mathbf{K}))(p_j - \omega^f_{j,t}(\mathbf{K})) - \sum_{j' \notin \mathcal{J}_f} s_{j',t}(\mathbf{p}^*_t(\mathbf{K}))\omega^f_{j',t}(\mathbf{K}).$$

Hence, the equilibrium prices in period t correspond to equilibria of the stage game where each firm f simultaneously chooses prices to maximize

$$\sum_{j \in \mathcal{J}_f} s_{j,t}(\mathbf{p})(p_j - \omega^f_{j,t}(\mathbf{K})) - \sum_{j' \notin \mathcal{J}_f} s_{j',t}(\mathbf{p})\omega^f_{j',t}(\mathbf{K}).$$

A.2 Analysis of Oligopoly Market

We follow closely the structue of Section 2.3 and state the generalized results here.

A.2.1 Equilibrium Existence, Uniqueness, and Continuity

Analogously to Equation 6, we define

$$g_f(\mathbf{p}) := \underbrace{\left(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p})\right)^{-1} D_{\mathbf{p}^f}\left(\mathbf{s}(\mathbf{p})^\intercal \boldsymbol{\omega}^f\right)}_{\substack{\text{net opportunity costs} \\ \text{of selling}}} - \underbrace{\left(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p})\right)^{-1}\mathbf{s}^f(\mathbf{p})}_{\substack{\text{inverse quasi} \\ \text{own-price elasticities}}} \tag{9}$$

Then, we can generalize Assumption 3 and Lemma 2 as follows.

General Assumption 3. *The following two conditions hold,*

i) $D_{\mathbf{p}_f} g_f(\mathbf{p}) - 1 \neq 0$ *for all* $\mathbf{p}$ *and* f*;*

ii) $\det\left(D_{\mathbf{p}}\big(\mathbf{g}(\mathbf{p})\big) - I \right) \neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p}) := (g_f(\mathbf{p}))_{f\in\mathscr{F}}$*.*

General Lemma 2. *Let Assumptions 1, 2, and General Assumption 3 hold. Then, the stage game admits a unique equilibrium.*

A.2.2 Continuity of Equilibrium Prices in Scarcity Effect Matrix Ω

General Lemma 3. *If the equilibrium of the stage game is unique for a compact set* $\mathscr{O}$ *of costs* Ω*, then there exists an equilibrium price vector* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *that is continuous in* Ω *on* $\mathscr{O}$ *and* $\boldsymbol{\theta}$ *on* Θ*.*

A.2.3 Characterization of Continuous-time Limit

General Proposition 2 (Continuous-time limit Limit)**.** *Let Assumptions 1, 2, and General Assumption 3 hold for* $\Omega = \mathbf{0}$*. For every* $\mathbf{K}$*, there exists a* $T_0(\mathbf{K}) > 0$*, non-increasing in* $\mathbf{K}$*, so that for any* $T \leq T_0(\mathbf{K})$ *there exists a unique equilibrium of the dynamic pricing game for sufficiently small* Δ*. Then, there exists a unique subgame-perfect equilibrium. The value function* $\Pi_{f,t}(\mathbf{K};\Delta)$ *converges to a limit* $\Pi_{f,t}(\mathbf{K})$ *that solves the differential equation*

$$\begin{aligned}\dot{\Pi}_{f,t}(\mathbf{K}) = -\lambda_t \quad \Bigg(& \sum_{j\in\mathscr{J}_f} s_j(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t))\big(p_j^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t) - (\Pi_t^f(\mathbf{K}) - \Pi_t^f(\mathbf{K}-\mathbf{e}_j))\big) \\ & - \sum_{j'\neq\mathscr{J}_f} s_{j'}(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}))\big(\Pi_t^f(\mathbf{K}) - \Pi_t^f(\mathbf{K}-\mathbf{e}_{j'})\big)\Bigg)\end{aligned}$$

with boundary conditions $\Pi_t^f(\mathbf{K}) = \mathbf{0}$ *if* $K_j = 0$ *for all* $j \in \mathscr{J}_f$ *and* $\Pi_T^f(\mathbf{K}) = 0$*. (i)* $\Pi_{f,T}(\mathbf{K}) = 0$*, (ii)* $\Pi_{f,t}(\mathbf{K}) = 0$ *if* $K_j = 0$ *for all* $j \in \mathscr{J}_f$*, (iii)* $\Pi_{f,t}(\mathbf{K};) = -\infty$ *if* $K_j < 0$ *for a* $j \in \mathscr{J}_f$*, and (iv)* $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j) = \Pi_{f,t}(\mathbf{K})$ *if* $K_j = 0$ *for a* $j \notin \mathscr{J}_f$*,* $K_{j'} \geq 0$ *for all* $j' \in \mathscr{J}_f$*.*

Lemma 4. *For a logit demand system as defined in Equation 3, holding everything else fixed, there exists a* $\bar{\sigma}$ *and a* $\bar{\Delta} > 0$ *so that for all* $\sigma > \bar{\sigma}$ *and* $\Delta < \bar{\Delta}$*, the cost matrix* $\Omega_t(\mathbf{K})$ *satisfies Assumption 4 for all* $t \in [0, T]$ *and* $\mathbf{K} \leq \mathbf{K}_0$.

A.2.4 Additional Theoretical Results on Dynamic Price Competition

Capacity Distribution and Prices Assume that λ_t and $s_{f,t}$ is independent of time, i.e., $\lambda_t = \lambda$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}$.

General Proposition 3. *For* $\mathbf{K}$ *with* $\underline{K} := \min_j K_j$*, the following holds:*

$$p_{j,t}(\mathbf{K}) = p^*_{j,T} + \mathcal{O}(|T-t|^{\underline{K}}),\ t \to T.$$

If $(\Pi^f_t)^{(\underline{K})}(\mathbf{K} - \mathbf{e}_{j'}) \neq 0$ *for all* f *and* j' *with* $K_{j'} = \underline{K}$*, then*

$$p_{j,t}(\mathbf{K}) = p^*_{j,T} + \Theta(|T-t|^{\underline{K}}),\ t \to T.$$

The proposition shows that prices are more different from 0 close to the deadline the smaller the minimum inventory of products $K_j = \underline{K}$ is. If firms have the same capacity, then any sale leads to price jump. This leads to strong incentives to get out of this state by offering low prices — possibly even prices smaller than the competitive price.

Independence of Irrelevant Alternatives and Markup Formula Assumption 4 generalizes to more than two products as follows.

General Assumption 4 (Independence of Irrelevant Alternatives (IIA))**.** $\frac{\partial}{\partial p_j}\frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})} = 0$ *for* $j \neq j_1, j_2$.

Given Assumptions 1, 2 and General Assumption 4, we can show that the game with multi-product firms can be transformed into a game of single-product firms.

General Proposition 4 (Mark-up formula under IIA)**.** *Let Assumptions 1, 2 and 4 hold. Then, there exists and equilibrium* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *of the above stage game for any scarcity matrix*

Ω. *Further, there exist functions $d_j(\mathbf{p}_{-f};\Omega,\boldsymbol{\theta})$ so that the equilibrium prices of the stage game coincide with the equilibrium prices of a game with a set $\mathscr{J}$ of players who each simultaneously choose a price p_j maximizing*

$$s_j(\mathbf{p})(p_j - c_j(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta})) + d_j(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta})$$

with a cost function

$$c_j(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta}) := \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j})\omega_{j'}^f \tag{10}$$

and $\tilde{s}_{j,j'}(\mathbf{p}_{-j}) := \frac{\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p})}{\sum_{j'' \in \mathscr{J}^0 \setminus \{j\}} \frac{\partial s_{j''}}{\partial p_j}(\mathbf{p})}$.

A consequence of Proposition 4 is that the first-order conditions (FOCs) that implicitly define the best response functions of the firms, can be written in a markup formulation as

$$\frac{p_j - c_j(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta})}{p_j} = -\frac{1}{\epsilon_j(\mathbf{p})} \tag{11}$$

where $\epsilon_j(\mathbf{p}) = \frac{\partial s_j(\mathbf{p})}{\partial p_j}\frac{p_j}{s_j(\mathbf{p})}$ is the elasticity of demand. The formulation (11) emphasizes the impact of the competitive forces in the presence of opportunity costs: Other firm's prices do not only impact own demand, but also the effective cost of selling the product.

B Proofs

B.1 Technical results

B.1.1 Continuous time limit

We use the following convergence result as $\Delta \to 0$ for the proofs of Lemma 1 and General Proposition 2.

Lemma 5. *Consider a continuous price function* $(\Omega,\boldsymbol{\theta})\mapsto\mathbf{p}^*(\Omega,\boldsymbol{\theta})=(p_f^*(\Omega,\boldsymbol{\theta}))_f$ *on a compact set* $\mathcal{O}$*, and a bounded and continuous function* $A:\mathbb{R}^{\mathcal{F}}\times\mathbb{R}^{\mathcal{J}\times\mathcal{F}}$*. Let* $\Pi_{f,t}(\mathbf{K};\Delta)$*,* $f\in\mathcal{F}$*, be a solution to the difference equations*

$$\left(\frac{\Pi_{f,t+\Delta}(\mathbf{K};\Delta)-\Pi_{f,t}(\mathbf{K};\Delta)}{\Delta}\right)_f=-\lambda_t\mathbf{A}\Big(\mathbf{p}^*\big(\Omega(\mathbf{K};\Delta)),\boldsymbol{\theta}_t\big),\,\Omega(\mathbf{K};\Delta)\Big)$$

where $\Omega(\mathbf{K};\Delta)=(\omega^f_{j,t}(\mathbf{K};\Delta))_{f,j}$*,* $\omega^f_{j,t}(\mathbf{K};\Delta):=\boldsymbol{\Pi}_{f,t+\Delta}(\mathbf{K};\Delta)-\boldsymbol{\Pi}_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)$*, with boundary conditions (i)* $\Pi_{f,T+\Delta}(\mathbf{K};\Delta)=0$*, (ii)* $\Pi_{f,t}(\mathbf{K};\Delta)=0$ *if* $K_j=0$ *for all* $j\in\mathcal{J}_f$*, (iii)* $\Pi_{f,t}(\mathbf{K};\Delta)=-\infty$ *if* $K_j<0$ *for a* $j\in\mathcal{J}_f$*, and (iv)* $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j;\Delta)=\Pi_{f,t}(\mathbf{K};\Delta)$ *if* $K_j=0$ *for a* $j\notin\mathcal{J}_f$*,* $K_{j'}\geq 0$ *for all* $j'\in\mathcal{J}_f$*. Then,* $(\Pi^f_t(\mathbf{K};\Delta))_f$ *converges to a limit* $(\Pi_{f,t}(\mathbf{K}))_f$ *that satisfies*

$$\big(\dot{\Pi}_{f,t}(\mathbf{K})\big)_f=-\lambda_t\mathbf{A}\Big(\mathbf{p}^*\big(\Omega(\mathbf{K}),\boldsymbol{\theta}_t\big),\,\Omega(\mathbf{K})\Big),$$

where $\Omega(\mathbf{K})=(\omega^f_{j,t}(\mathbf{K}))_{f,j}$*,* $\omega^f_{j,t}(\mathbf{K}):=\boldsymbol{\Pi}_{f,t}(\mathbf{K};\Delta)-\boldsymbol{\Pi}_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j)$*, with boundary conditions (i)* $\Pi_{f,T}(\mathbf{K})=0$*, (ii)* $\Pi_{f,t}(\mathbf{K})=0$ *if* $K_j=0$ *for all* $j\in\mathcal{J}_f$*, (iii)* $\Pi_{f,t}(\mathbf{K};)=-\infty$ *if* $K_j<0$ *for a* $j\in\mathcal{J}_f$*, and (iv)* $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j)=\Pi_{f,t}(\mathbf{K})$ *if* $K_j=0$ *for a* $j\notin\mathcal{J}_f$*,* $K_{j'}\geq 0$ *for all* $j'\in\mathcal{J}_f$*.*

Proof. Since $\mathbf{A}$ is bounded, the difference equations show that $(\Pi_f(\mathbf{K};\Delta))_{f\in\mathcal{F},\mathbf{K}\leq\mathbf{K}_0}$ is equicontinuous and equibounded in t as $\Delta\to 0$. Hence, by the Arzela-Ascoli Theorem, there exist limit points $(\Pi_f(\mathbf{K}))_{f\in\mathcal{F},\mathbf{K}\leq\mathbf{K}_0}$. We claim that

$$\big(\Pi_{f,t}(\mathbf{K})\big)_f=\int_t^T\lambda_u\mathbf{A}\Big(\mathbf{p}^*(\big(\Omega_u(\mathbf{K}),\boldsymbol{\theta}_u\big),\,\Omega_u(\mathbf{K})\Big)du. \tag{12}$$

To this end, we note that if we let $\lceil u\rceil_\Delta$ to be the largest number that is divisible by Δ and smaller or equal than u

$$\big(\Pi_{f,t}(\mathbf{K};\Delta)\big)_f=\int_t^T\lambda_{\lceil u\rceil_\Delta}\mathbf{A}\Big(\mathbf{p}^*\big(\Omega_{\lceil u\rceil_\Delta}(\mathbf{K};\Delta),\alpha_{\lceil u\rceil_\Delta}\big),\,\Omega_{\lceil u\rceil_\Delta}(\mathbf{K};\Delta)\Big)du. \tag{13}$$

We take the limit $\Delta \to 0$ on both sides. The left-hand side of (13) converges to the left-hand side of (12). On the right-hand side, $\Omega_{\lceil u \rceil_\Delta}(\mathbf{K};\Delta)$ converges to $\Omega_u(\mathbf{K})$. Hence, by continuity of $\mathbf{p}^*$ and $\mathbf{A}$ the integrand in (13) converges to the integrand in (12). The dominated convergence theorem finishes the proof. ■

B.1.2 Continuity of stage game prices

Lemma 6. *Let $\mathscr{P} \subset \mathbb{R}^{\mathscr{J}}$ be compact and convex and $\mathscr{O}$ a path-connected set of $(\Omega, \boldsymbol{\theta})$. Further, let $g : (\mathbf{q}; \Omega, \boldsymbol{\theta}) \mapsto \mathbf{p}$ be defined as a function $\mathscr{P} \times \mathscr{O} \to \mathscr{P}$, where g is continuously differentiable in $\mathbf{q}$ and continuous in Ω and $\boldsymbol{\theta}$. If $\det(D_{\mathbf{q}} g(\mathbf{q}; \Omega, \boldsymbol{\theta}) - I) \neq 0$ for all $(\mathbf{q}; \Omega, \boldsymbol{\theta}) \in \mathscr{P} \times \Theta$, then there is a unique $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ satisfying $g(\mathbf{p}^*(\boldsymbol{\theta}); \Omega, \boldsymbol{\theta}) = \mathbf{p}^*(\Omega, \boldsymbol{\theta})$ and it depends continuously on Ω and $\boldsymbol{\theta}$.*

Proof. The existence and uniqueness of $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ follows directly from Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010). To show continuity, we consider a sequence $(\Omega_n, \boldsymbol{\theta}_n)_{n \geq 1}$ converging to some $(\Omega_\infty, \boldsymbol{\theta}_\infty)$. Thanks to path-connectedness of $\mathscr{O}$ there exists a continuous path $\mathbf{r} : [0,1] \to \mathscr{O}$ and a sequence $a_n \uparrow 1$ such that $\mathbf{r}(a_n) = (\Omega_n, \boldsymbol{\theta}_n)$ and $\mathbf{r}(1) = (\Omega_\infty, \boldsymbol{\theta}_\infty)$. By Browder's Theorem (Theorem 1.1 in Solan and Solan (2021)), the set $\{(\mathbf{p}^*(\mathbf{r}(a)); a) : a \in [0,1]\} \subset \mathscr{P} \times [0,1]$ is connected. By the main theorem of connectedness, each set $\{(p_j^*(\mathbf{r}(a)); a) : a \in [0,1]\} \subset \mathbb{R} \times [0,1]$ is connected, for all j. By Burgess (1990), the function $a \mapsto p_j^*(\mathbf{r}(a))$ is continuous, so $p_j^*(\Omega_n, \boldsymbol{\theta}_n) = p_j^*(\mathbf{r}(a_n)) \to p_j^*(\mathbf{r}(1)) = p_j^*(\Omega_\infty, \boldsymbol{\theta}_\infty)$. ■

B.2 Proofs of Single Firm Model

B.2.1 Proof of Lemma 1

The profit-maximizing prices of the stage game $\mathbf{p}_M(\boldsymbol{\omega})$ are implicitly given by (4).

$$g(\mathbf{q}; \boldsymbol{\omega}, \boldsymbol{\theta}) := \boldsymbol{\omega} - \underbrace{(D_{\mathbf{p}} \mathbf{s}_t(\mathbf{q}))^{-1} \mathbf{s}_t(\mathbf{q})}_{\leq 0}$$

is continuously differentiable in $\mathbf{q}$, Ω and $\boldsymbol{\theta}$ by Assumption 1, and any fixed point must satisfy $\mathbf{q} \geq \boldsymbol{\omega}$ and $\mathbf{q} \leq \boldsymbol{\omega} + \mathbf{1}\bar{\epsilon}$ by Assumption **??** iii). Hence, the convergence to 4 follows by Lemma 5.

B.2.2 Proof of Proposition 1

Proof. i) To see that $\Pi_{M,t}$ is decreasing in t, note that in (4), p_j can always be chosen so that objective function in the maximum is positive. Hence, $\dot{\Pi}_t^M(\mathbf{K}) < 0$.

Next, we show that $\Pi_t^M(\boldsymbol{K}) > \Pi_t^M(\boldsymbol{K} - \boldsymbol{e}_j)$ for all j by induction in $\sum_j K_j$.

Induction start: It is immediate that $\Pi_t^M(\mathbf{e}_j) \geq \Pi_t^M(\mathbf{0}) = 0$ for all j and $t \leq T$.

Induction hypothesis: Assume that $\Pi_t^M(\boldsymbol{K}) > \Pi_t^M(\boldsymbol{K} - \mathbf{e}_j)$ for all $\mathbf{K}$ such that $\sum_j K_j = \bar{K}$.

Induction step: Now, consider a capacity vector $\mathbf{K}$ with $\sum_j K_j = \bar{K} + 1$. By sub-optimality of the prices $\mathbf{p}^M(\boldsymbol{\omega}_t^M(\mathbf{K} - \mathbf{e}_{\bar{j}}))$ given capacity vector $\mathbf{K}$, we have

$$\Pi_t^M\big(\mathbf{K}\big) \geq \int_t^T \lambda_z \Big[\sum_j s_{j,z}\big(\mathbf{p}^M(\boldsymbol{\omega}_z^M(\mathbf{K} - \mathbf{e}_{\bar{j}}))\big)\big(p_{j,z}^M(\boldsymbol{\omega}_z^M(\mathbf{K} - \mathbf{e}_{\bar{j}})) + \Pi_z^M(\mathbf{K} - \mathbf{e}_j)\big) \cdot e^{-\int_t^z \lambda_u \sum_{j''} s_{j'',u}(\mathbf{p}_u^M(\boldsymbol{\omega}_t^M(\mathbf{K})))du} dz > \Pi_t^M\big(\mathbf{K} - \mathbf{e}_{\bar{j}}\big)$$

where the last inequality follows form $\Pi_z^M(\mathbf{K} - \mathbf{e}_j) > \Pi_z^M(\mathbf{K} - \mathbf{e}_j - \mathbf{e}_{\bar{j}})$ by the induction hypothesis.

ii) Next, we show that $\Pi_t^M(\mathbf{K}) - \Pi_t^M(\mathbf{K} - \mathbf{e}_j) \leq \Pi_t^M(\mathbf{K} - \mathbf{e}_j) - \Pi_t^M(\mathbf{K} - 2\mathbf{e}_j)$ for all j. To this end, let

$$H(\mathbf{x}; \boldsymbol{\theta}) = -\max_{\mathbf{p}} \sum_j s_j(\mathbf{p}; \boldsymbol{\theta})(p_j - x_j).$$

Note that H is concave as a minimum of affine functions, strictly increasing in $\mathbf{x}$, and

$H(\mathbf{0};\boldsymbol{\theta})=0$ by Assumption **??** iii). Since H is concave, it admits the representation

$$H(\mathbf{x};\boldsymbol{\theta})=\inf_{\mathbf{s}}(\mathbf{s}\cdot\mathbf{x}-H^*(\mathbf{s};\boldsymbol{\theta}))$$

where the concave $H^*(\mathbf{s};\boldsymbol{\theta})=\inf_{\mathbf{x}}(\mathbf{x}\cdot\mathbf{s}-H(\mathbf{x};\boldsymbol{\theta}))$ is the concave conjugate of H, with $H^*(\mathbf{0};\boldsymbol{\theta})=0$. Moreover,

$$\dot{\Pi}_t^M(\mathbf{K})=\lambda_t H(\nabla\Pi_t(\mathbf{K});\boldsymbol{\theta}_t)$$

where $\nabla\Pi_t^M(\mathbf{K})=\left(\Pi_t^M(\mathbf{K})-\Pi_t^M(\mathbf{K}-\mathbf{e}_j)\right)_j$. Thus, $\Pi_t^M(\mathbf{K})$ is the value function for the optimal control problem

$$\Pi_t^M(\mathbf{K})=\sup_{\mathbf{s}\in\mathscr{A}}\mathbb{E}\left[\int_t^T\lambda_u H^*(\mathbf{s}_u;\boldsymbol{\theta}_u)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}}=\mathbf{K}\right]=:\sup_{\mathbf{s}}J_t(\mathbf{K},\mathbf{s})$$

where $\mathbf{X}_t^{\mathbf{a}}$ is the process which jumps by $-\mathbf{e}_j$ at rate $\lambda_t s_{j,t}$ and $\mathbf{s}\in\mathscr{A}$ are processes adapted with respect to $\mathbf{X}^{\mathbf{s}}$, with the property $s_{j,t}=0$ if $X_{j,t}^{\mathbf{s}}=0$ (Theorem 8.1 in Fleming and Soner (2006)). Let $\mathbf{s}_{\mathbf{K}}^*$ be the optimal control in the previous equation and $\mathbf{s}_{K-2}^*$ be the optimal control when $\mathbf{K}$ is replaced by $\mathbf{K}-2\mathbf{e}_j$. Then, note that since $\mathbf{s}_{\mathbf{K}}^*,\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*\in\mathscr{A}$, $\frac{\mathbf{s}_{\mathbf{K}}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*}{2}\in\mathscr{A}$ because the process $\left(\mathbf{X}_s^{\frac{\mathbf{s}_{\mathbf{K}}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^*}{2}}\right)_s$ can be chosen as $\left(\frac{\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}}^*}+\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^*}}{2}\right)_s$ ("coupling argument"). Hence,

$$\begin{aligned}
&\Pi_t^M(\mathbf{K})+\Pi_t^M(\mathbf{K}-2\mathbf{e}_j)-2\Pi_t^M\left(\mathbf{K}-\mathbf{e}_j\right) &\le\\
&J_t(\mathbf{K},s_{\mathbf{K}}^*)+J_t(\mathbf{K}-2\mathbf{e}_j,s_{\mathbf{K}-2\mathbf{e}_j}^*)-2\,J_t\left(\mathbf{K}-\mathbf{e}_j,\frac{\mathbf{s}_{\mathbf{K}}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*}{2}\right) &\le\\
&\mathbb{E}\left[\int_t^T\lambda_u\left(H^*(\mathbf{s}_{\mathbf{K},u}^*)+H^*(\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j,u}^*)-2H^*\left(\frac{\mathbf{s}_{\mathbf{K},u}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f,u}^*}{2}\right)\right)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}_{\mathbf{K}}^*}=\mathbf{K},\,\mathbf{X}_t^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*}=\mathbf{K}-2\mathbf{e}_j,\right] &\le 0.
\end{aligned}$$

iii) To show that $\omega^M_{j,t}(\mathbf{K}_t)$ is a submartingale, we show that for any capacity vector $\mathbf{K}$,

$$\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)\middle|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}\geq 0.$$

To this end, first, note that $\mathbf{K}_t$ is right-continuous in t. Further, for $\mathbf{K}$ with $K_j=0$, we set $\omega^M_{j,t}(\mathbf{K})=\infty$ for all t. Thus, we are setting the opportunity cost of selling a unit if no capacity is left to infinity, which is equivalent to the constraint of not being able to sell units that are not available.

Then, we have for $\bar{\mathbf{K}}$ with $\bar{K}_j=1$ that

$$\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}>0.$$

Next consider $\bar{\mathbf{K}}$ with $\bar{K}_j\geq 0$. Then, we have that

$$\begin{aligned}&\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}=\\&\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_{t+\Delta})|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}+\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}=\\&\dot{\omega}^M_{j,t}(\bar{\mathbf{K}})+\lambda_t\sum_{j'}s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\left(\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\omega^M_{j,t}(\bar{\mathbf{K}})\right)\end{aligned}$$

by right-continuity of the process K_t. By (4), we can write

$$\dot{\omega}^M_{j,t}(\bar{\mathbf{K}})\;=\;-\lambda_t\left[\sum_{j'}s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\left(p^M_{j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}})\right)-s_{j',t}(p^M_t(\bar{\mathbf{K}}-\mathbf{e}_j))\left(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\right].$$

and we know that

$$\begin{aligned}-\omega^M_{j',t}(\bar{\mathbf{K}})+\omega^M_{j,t}(\bar{\mathbf{K}})-\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})&=\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_j)-\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'})+\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'}-\mathbf{e}_j)\\&=\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\end{aligned}$$

Hence, $\lim_{\Delta \to 0} \frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|_t = \bar{\mathbf{K}}\right]}{\Delta}$ is equal to

$$-\lambda_t\Big[\sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}})\big)\big(p^M_{j',t}(\bar{\mathbf{K}}) - \omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big) - s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\big(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j) - \omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\Big]$$

Then, note that by definition of $\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)$,

$$\sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}})\big)\big(p^M_{j',t}(\bar{\mathbf{K}}) - \omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\big) \leq \sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)\big)\big(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)) - \omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})\big).$$

Hence, $\lim_{\Delta \to 0} \frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|_t = \bar{\mathbf{K}}\right]}{\Delta} \geq 0$. ■

B.3 Proofs of General Oligopoly Model

B.3.1 Proof of General Lemma 2

First, we show that there exists a $\bar{p} < \infty$ so that for any any vector of prices $\mathbf{q}$, the best response price p_j for any product j is bounded by $\bar{p}$. We proceed with a proof by contradiction.

Assume that there is an increasing sequence of $\bar{b}^n \to_{n\to\infty}$ such that there is a vector of prices $\mathbf{q}^n$ such that there is a best response price $p^n_j > \bar{b}^n$.

$$
\begin{aligned}
0 \leq\quad & \underbrace{\frac{\partial s_j}{\partial p_j}}_{<0}(p_j-\omega_j^f)+\sum_{k\in\mathscr{J}_f\setminus\{j\}}\underbrace{\frac{\partial s_k}{\partial p_j}}_{>0}(p_k-\omega_k^f) \\
& -\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\omega_k^f+s_j(\mathbf{q}^{-f},\mathbf{p}^f) \\
\leq\quad & \left|\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\right|\left[\frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega_k^f)_+-\left(2(p_j-\omega_j^f)+\underbrace{\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}}_{\geq-\bar{\epsilon}\text{ by Assumption ??-??}}\right)\right. \\
& \left.+\left(\sum_{k\notin\mathscr{J}_f}\frac{|\omega_k^f|}{J-1}\right)\right].
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
& \forall\, j\in\mathscr{J}_f:\quad 2(p_j-\omega_j^f)+\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}\leq\frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega_k^f)_++\sum_{k\notin\mathscr{J}_f}\frac{|\omega_k^f|}{J-1} \\
\Rightarrow\quad & \forall\, j\in\mathscr{J}_f:\quad 2(p_j-\omega_j^f)-\bar{\epsilon}\leq\frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega_k^f)_++\sum_{k\notin\mathscr{J}_f}\frac{|\omega_k^f|}{J-1}.
\end{aligned}
$$

$$
\begin{aligned}
0 \leq\quad & \underbrace{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^{f,*}_{-f},p_j)}_{<0}(p_j-\omega_j^f)+\sum_{k\in\mathscr{J}_f\setminus\{j\}}\underbrace{\frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}_{>0}(p_k-\omega_k^f) \\
& -\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\omega_k^f+s_j(\mathbf{q}^{-f},\mathbf{p}^f) \\
\leq\quad & \left|\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\right|\left[\frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega_k^f)_+-\left(2(p_j-\omega_j^f)+\underbrace{\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}}_{\geq-\bar{\epsilon}\text{ by Assumption ??-??}}\right)\right. \\
& \left.+\left(\sum_{k\notin\mathscr{J}_f}\frac{|\omega_k^f|}{J-1}\right)\right].
\end{aligned}
$$

This is equivalent to

$$\begin{aligned}
&\forall\, j \in \mathscr{J}_f: \quad 2(p_j - \omega_j^f) + \frac{s_j(\mathbf{q}^{-f}, \mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f}, \mathbf{p}^f)} \leq \frac{1}{J-1} \sum_{k \in \mathscr{J}_f \setminus \{j\}} (p_k - \omega_k^f)_+ + \sum_{k \notin \mathscr{J}_f} \frac{|\omega_k^f|}{J-1} \\
\Rightarrow \quad &\forall\, j \in \mathscr{J}_f: \quad 2(p_j - \omega_j^f) - \bar{\epsilon} \leq \frac{1}{J-1} \sum_{k \in \mathscr{J}_f \setminus \{j\}} (p_k - \omega_k^f)_+ + \sum_{k \notin \mathscr{J}_f} \frac{|\omega_k^f|}{J-1}.
\end{aligned}$$

Using this for $j = l$ maximizing the left-hand side we obtain a contradiction once we choose $\bar{p}$ sufficiently large. Thus, the best response of each firm for each product is strictly smaller than a constant $\bar{p}$.

Hence, by Assumption **??**, there is a unique fixed point of $g(\mathbf{p})$ by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010).

B.3.2 Proof of General Lemma 3

B.3.3 Proof of General Proposition 2

B.3.4 Proof of Lemma 4

B.3.5 Proof of General Proposition 3

Assume that λ_t and $s_{f,t}$ is independent of time, i.e., $\lambda_t = \lambda$, $\alpha_t = \alpha$. For t close to T, we know from Lemma 2 that the equilibirum of the stage game is unique and the price vectors $p_t^*(\mathbf{K})$ are implicitly defined by a system of equations given by

$$D_{\mathbf{p}^f} \mathbf{s}^f(\mathbf{p}_t^*(\mathbf{K}))\mathbf{p}_t^{f,*}(\mathbf{K}) - \mathbf{s}(\mathbf{p}_t^*(\mathbf{K}))\boldsymbol{\omega}_t^f(\mathbf{K}) = 0$$

for all f. The only time-dependent variables are then $\Omega_t(\mathbf{K}) = (\boldsymbol{\omega}_t^f)_{f \in \mathscr{F}}$. The n-th time derivative $(p_t^*)^{(n)}(\mathbf{K})$ depends on the time derivatives $\Omega_t(\mathbf{K}), \ldots, \Omega_t^{(n)}(\mathbf{K})$.

We are interested in the limit as $t \to T$. First, $\lim_{t\to T} \Omega_t = 0$. Furthermore, we can write

$$\begin{aligned}\dot{\omega}^f_{j,t}(\mathbf{K}) =& \dot{\Pi}^f_t(\mathbf{K}) - \dot{\Pi}^f_t(\mathbf{K}-\mathbf{e}_j)\\ =& -\lambda\big[\mathbf{s}^f(\mathbf{p}^*_t(\mathbf{K}))\mathbf{p}^{f,*}_t(\mathbf{K}) - \mathbf{s}(\mathbf{p}^*_t(\mathbf{K}))\boldsymbol{\omega}^f_t(\mathbf{K}) - (\mathbf{s}^f(\mathbf{p}^*_t(\mathbf{K}-\mathbf{e}_j))\mathbf{p}^{f,*}_t(\mathbf{K}-\mathbf{e}_j) - \mathbf{s}(\mathbf{p}^*_t(\mathbf{K}-\mathbf{e}_j))\boldsymbol{\omega}^f_t(\mathbf{K}-\mathbf{e}_j))\big]\end{aligned}$$

Thus, as $t \to T$, $\dot{\omega}^f_{j,t}(\mathbf{K}) = 0$ if $K_j > 1$. If $j \in \mathscr{J}_f$ and $K_j = 1$, then $\dot{\omega}^f_{j,t}(\mathbf{K}) < 0$. If $j \notin \mathscr{J}^f$ and $K_j = 1$, then by the competition effect $\dot{\omega}^f_{j,t}(\mathbf{K}) > 0$.

This implies that $\dot{p}^*_{j,T}(\mathbf{K}) < 0$ if $K_j = 1$ and $\dot{p}^*_{j,T}(\mathbf{K}) = 0$ otherwise.

Induction assumption: If $K_j > n-1$ for all j, then as $t \to T$, $(\omega^f_{j,t})^{(n-1)}(\mathbf{K}) = 0$ for all f, j.

We can also calculate all other time derivatives recursively

$$(\omega^f_{j,t})^{(n)}(\mathbf{K}) = -\lambda\big[G^n((\Omega^{(m)}_t(\mathbf{K}))^{n-1}_{m=0}) - G^n(\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j))^{n-1}_{m=0}))\big].$$

Then, note if $\min_i K_i > n$, then $(\omega^f_{j,t})^{(n)}(\mathbf{K}) = 0$. If $\min_i K_i = n$, then

$$(\omega^f_{j,t})^{(n)}(\mathbf{K}) = -\lambda\big[-G^{(n)}(\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j))^{n-1}_{m=0}))\big] = -(\Pi^f_t)^{(n)}(\mathbf{K}-\mathbf{e}_j).$$

B.3.6 Proof of General Proposition 4

Equivalence Note that the profit-maximizing prices of the stage game $\mathbf{p}^M(\boldsymbol{\omega};\boldsymbol{\theta})$ are implicitly given by (4) and

$$g(\mathbf{q};\boldsymbol{\theta}) := \boldsymbol{\omega} - \underbrace{(D_{\mathbf{p}}\mathbf{s}_t(\mathbf{q}))^{-1}\mathbf{s}_t(\mathbf{q})}_{\leq 0}$$

is continuously differentiable in $\mathbf{q}$ and $\boldsymbol{\theta}$ by Assumption 1 i), and any fixed point must satisfy $\mathbf{q} \geq \boldsymbol{\omega}$ and $\mathbf{q} \leq \boldsymbol{\omega} + \mathbf{1}\bar{\epsilon}$ by Assumption **??** iii). Hence, the convergence to 4 follows by Lemma 5.

Recall that the first-order conditions of firm f's payoff with respect to product $j \in \mathscr{J}_f$

are given by

$$p_j - \left(\omega_j^f - \sum_{j' \in \mathcal{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathcal{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \right) = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

Further, the observation that $\frac{\partial s_j(\mathbf{p})}{\partial p_j} = -\sum_{j'' \in \mathcal{J}^0 \setminus \{j\}} \frac{\partial s_{j''}(\mathbf{p})}{\partial p_j}$, and by Assumption **??** (Independence of Irrelevant alternatives) it follows that

$$\begin{aligned}
c(\mathbf{p}_{-j};\boldsymbol{\omega}) :=& \omega_j^f - \sum_{j' \in \mathcal{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathcal{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \\
=& \omega_j^f - \sum_{j' \in \mathcal{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\sum_{j' \in \mathcal{J}^0 \setminus \{j\}} \frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathcal{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\sum_{j'' \in \mathcal{J}^0 \setminus \{j\}} \frac{\partial s_{j''}(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \\
=& \omega_j^f - \sum_{j' \in \mathcal{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j' \notin \mathcal{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j}) \omega_{j'}^f.
\end{aligned}$$

Thus, the first-order conditions of the stage game are equivalent to the first order conditions of a game with $\mathcal{J}$ players where each player j's payoff is given by

$$s_j(\mathbf{p})(p_j - c(\mathbf{p}_{-j};\boldsymbol{\omega})) + d(\mathbf{p}_{-j};\boldsymbol{\omega}).$$

Existence Assume $s_j(\mathbf{p};\boldsymbol{\theta}) > 0$ satisfies Assumptions **??** and **??**. Then, we define the best-response function of "player" j in the game defined in Lemma 4 by

$$\mathcal{R}: \quad \mathbf{q} \mapsto \left(\arg\max_{p_j} s_j(\mathbf{q})(p_j - c_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta})) + d_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) \right)_{j \in \mathcal{J}}.$$

where for $\tilde{s}_{j,j'}(\mathbf{q}_{-j}) := \frac{\frac{\partial s_{j'}}{\partial p_j}(\mathbf{q})}{\sum_{j'' \in \mathcal{J}^0 \setminus \{j\}} \frac{\partial s_{j''}}{\partial p_j}(\mathbf{q})}$ and $j \in \mathcal{J}_f$

$$c_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) := \omega_j^f - \sum_{j' \in \mathcal{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{q}_{-j})(q_{j'} - \omega_j^f) + \sum_{j' \notin \mathcal{J}_f} \tilde{s}_{j,j'}(\mathbf{q}_{-j}) \omega_{j'}^f. \tag{14}$$

First, we show that $\mathscr{R}$ is well-defined as a function $\mathbb{R}^{\mathscr{J}} \mapsto [-\infty, \infty]^{\mathscr{J}}$ (rather than a correspondence). To this end, note that player j's profit is increasing in p_j if and only if

$$p_j - c_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) + \underbrace{\frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{<0 \text{ by Assumption } \mathbf{??}} \leq 0 \tag{15}$$

and the left-hand side is increasing in p_j by Assumption **??**.

Then, note that the best-response function $\mathscr{R}$ takes values in $[\underline{p},\overline{p}]^{\mathscr{J}}$, with $\underline{p} > -\infty$ and $\overline{p} < \infty$, for all $\mathbf{q}$ by the same argument as in the proof of Lemma 2.

Now, consider $\mathscr{R} : [\underline{p},\overline{p}]^{\mathscr{J}} \to [\underline{p},\overline{p}]^{\mathscr{J}}$. In order to show continuitiy of $\mathscr{R}$, we use the implicit function theorem in the form of Theorem 1.A.4 in Dontchev and Rockafellar (2009). To this end, for $\epsilon > 0$, consider the mapping

$$\Phi : (\mathbf{p},\mathbf{q}) \mapsto \left(p_j - \epsilon\Big(p_j - c_j(\mathbf{q}_{-j};\Omega,\boldsymbol{\theta}) + \frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}\Big)\right)_{j\in\mathscr{J}}$$

Then $D_{\mathbf{p}}\Phi$ is a diagonal matrix with diagonal entries

$$\phi_j := 1 - \epsilon\Bigg(1 + \underbrace{\frac{\partial}{\partial p_j}\frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{\geq 0 \text{ by Assumption } \mathbf{??}}\Bigg)$$

Let $\epsilon > 0$ be so that $\phi_j > 0$ for all j. Then all diagonal entries are in $(0, 1-\epsilon)$ and Φ is Lipschitz continuous with Lipschitz constant $\max_j \phi_j$. Further $D_{\mathbf{q}}\Phi$ is bounded because it is continuous and the function is defined on a compact set $[\underline{p},\overline{p}]^{\mathscr{J}}$. Thus, $\mathscr{R}$ is continuous. Hence, by Brouwer's fixed-point theorem $\mathscr{R} : [\underline{p},\overline{p}]^{\mathscr{J}} \to [\underline{p},\overline{p}]^{\mathscr{J}}$ has a fixed point.

C Simple Logit and Nested Logit Calculations

C.1 Simple Logit Demand

Consider a logit logit demand system as specified in Equation 3: $s_j(\mathbf{p}) = \frac{e^{\frac{\delta_j - \alpha p_j}{\rho}}}{1+\sum_{j\in\mathcal{J}} e^{\frac{\delta_j-\alpha p_j}{\rho}}}$. Throughout this section we omit the arguments of the demand functions when there is no ambiguity. First, note that

$$\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{\rho} s_j(1-s_j) \qquad \frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{\rho} s_j s_{j'}.$$

First, we show that Assumption 1 is satisfied.

i) $\lim_{p_{j'}\to\infty, j'\notin\mathcal{A}}$

First, we show that Assumption 2 is satisfied. To this end, note that

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\rho}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \dots & s_1 s_J \\ s_2 s_1 & \ddots & & s_2 s_J \\ \vdots & & \ddots & \vdots \\ s_J s_1 & \dots & s_J s_{J-1} & -s_J(1-s_J) \end{pmatrix}^{-1} = -\frac{\rho}{\alpha s_0}\begin{pmatrix} 1+\frac{s_0}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+\frac{s_0}{s_J} \end{pmatrix}.$$

Hence,

$$\hat{\boldsymbol{\epsilon}} = (D_p\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\frac{\rho}{\alpha s_0}\begin{pmatrix} 1+\frac{s_0}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+\frac{s_0}{s_J} \end{pmatrix}\begin{pmatrix} s_1 \\ \vdots \\ s_J \end{pmatrix} = -\frac{\rho}{\alpha s_0}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and since $\frac{\partial}{\partial p_j}\frac{1}{s_0} = -\frac{\alpha}{\rho}\frac{s_j}{s_0}$,

$$\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}} - I\right) = \det\begin{pmatrix} -\frac{s_1}{s_0}-1 & \dots & -\frac{s_J}{s_0} \\ \vdots & \ddots & \vdots \\ -\frac{s_1}{s_0} & \dots & -\frac{s_J}{s_0}-1 \end{pmatrix} = (-1)^J\frac{1}{s_0}$$

Next, note that for a for a any $f \in \mathscr{F}$, if we define $s_0^f(\mathbf{p}) = 1 - \sum_{j \in \mathscr{J}_f} s_j(\mathbf{p})$, then

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\rho}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \dots & s_1 s_{J_f} \\ s_2 s_1 & \ddots & & s_2 s_{J_f} \\ \vdots & & \ddots & \vdots \\ s_{J_f} s_1 & \dots & s_{J_f} s_{J_f-1} & -s_{J_f}(1-s_{J_f}) \end{pmatrix}^{-1} = -\frac{\rho}{\alpha s_0^f}\begin{pmatrix} 1+\frac{s_0^f}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+\frac{s_0^f}{s_1} \end{pmatrix}.$$

Further, $\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}^f = \sum_j s_j \omega_j^f$ and

$$D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}^f) = \frac{\alpha}{\rho}\begin{pmatrix} s_1\big(\sum_{j \in \mathscr{J}} s_j \omega_j^f - \omega_1^f\big) \\ \vdots \\ s_{J_f}\big(\sum_{j \in \mathscr{J}} s_j \omega_j^f - \omega_{J_f}^f\big) \end{pmatrix}$$

Hence,

$$\begin{aligned}(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}) &= -\frac{1}{s_0^f}\begin{pmatrix} \big(\sum_{j \in \mathscr{J}_f} s_j + s_0^f\big)\sum_{j \in \mathscr{J}} s_j \omega_j^f - \sum_{j \in \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_1^f \\ \vdots \\ \big(\sum_{j \in \mathscr{J}_f} s_j + s_0^f\big)\sum_{j \in \mathscr{J}} s_j \omega_j^f - \sum_{j \in \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_{J_f}^f \end{pmatrix} \\ &= -\frac{1}{s_0^f}\begin{pmatrix} \sum_{j \in \mathscr{J} \setminus \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_1^f \\ \vdots \\ \sum_{j \in \mathscr{J} \setminus \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_{J_f}^f \end{pmatrix}\end{aligned}$$

Further, since

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}) = -\frac{\rho}{\alpha s_0^f}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

Finally,

$$g^f(\mathbf{p}) = -\begin{pmatrix}\frac{\sum_{j\in\mathcal{J}\setminus\mathcal{J}_f} s_j}{1-\sum_{j\in\mathcal{J}_f} s_j}\omega_j^f - \omega_1^f - \frac{1}{1-\sum_{j\in\mathcal{J}_f} s_j}\frac{\rho}{\alpha}\\ \vdots\\ \frac{\sum_{j\in\mathcal{J}\setminus\mathcal{J}_f} s_j}{1-\sum_{j\in\mathcal{J}_f} s_j}\omega_j^f - \omega_{J_f}^f - \frac{1}{1-\sum_{j\in\mathcal{J}_f} s_j}\frac{\rho}{\alpha}\end{pmatrix}$$

and

$$D_{\mathbf{p}}g^f(\mathbf{p}) = \left(\begin{pmatrix}-\frac{s_k}{1-\sum_{j\in\mathcal{J}_f} s_j}\\ \vdots\\ -\frac{s_k}{1-\sum_{j\in\mathcal{J}_f} s_j}\end{pmatrix}_{k\in\mathcal{J}_f}, \begin{pmatrix}\frac{\alpha}{\sigma}\frac{-(1-\sum_{j\in\mathcal{J}_f} s_j)(1-\sum_{j\notin\mathcal{J}_f} s_j)+(\sum_{j\in\mathcal{J}_f} s_j)(\sum_{j\notin\mathcal{J}_f} s_j)}{(1-\sum_{j\in\mathcal{J}_f} s_j)^2}\omega_j^f + s_k\frac{\sum_{j\in\mathcal{J}_f} s_j}{(1-\sum_{j\in\mathcal{J}_f} s_j)^2}\\ \vdots\\ \frac{\alpha}{\sigma}\frac{-(1-\sum_{j\in\mathcal{J}_f} s_j)(1-\sum_{j\notin\mathcal{J}_f} s_j)+(\sum_{j\in\mathcal{J}_f} s_j)(\sum_{j\notin\mathcal{J}_f} s_j)}{(1-\sum_{j\in\mathcal{J}_f} s_j)^2}\omega_j^f + s_k\frac{\sum_{j\in\mathcal{J}_f} s_j}{(1-\sum_{j\in\mathcal{J}_f} s_j)^2}\end{pmatrix}_{k\notin\mathcal{J}_f}\right)$$

For large σ the term in front of ω_j^f vanishes relative to the probability.

C.2 Nested logit demand

In this section, we consider a nested logit demand model given by

$$s_j(\mathbf{p}) = \underbrace{\frac{e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}{\sum_{j\in\mathcal{J}} e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}}_{=:s_{j|\mathcal{J}}(\mathbf{p})}\frac{\left(\sum_{i\in\mathcal{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}}{1+\left(\sum_{i\in\mathcal{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}} \qquad s_0(\mathbf{p}) = \frac{1}{1+\left(\sum_{i\in\mathcal{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}}$$

To simplify notation, let $D_J := \sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}$. Then,

$$\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{1-\sigma} s_j \left(1-\left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right)\right) \qquad \frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{1-\sigma} s_{j'} \left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right).$$

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = -\frac{1}{\alpha s_0}\begin{pmatrix} \frac{\sigma}{D_J^{1-\sigma}}+1+(1-\sigma)\frac{s_0}{s_1} & \frac{\sigma}{D_J^{1-\sigma}}+1 & \dots & \frac{\sigma}{D_J^{1-\sigma}}+1 \\ \frac{\sigma}{D_J^{1-\sigma}}+1 & \ddots & & \frac{\sigma}{D_J^{1-\sigma}}+1 \\ \vdots & & \ddots & \vdots \\ \frac{\sigma}{D_J^{1-\sigma}}+1 & \dots & \frac{\sigma}{D_J^{1-\sigma}}+1 & \frac{\sigma}{D_J^{1-\sigma}}+1+(1-\sigma)\frac{s_0}{s_j} \end{pmatrix}$$

Hence,

$$\hat{\boldsymbol{\epsilon}} = (D_p\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\left(\frac{\sigma}{\alpha s_0}+\frac{1-\sigma}{\alpha s_0}\right)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = -\frac{1}{\alpha s_0}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$

and therefore

$$D_p\hat{\boldsymbol{\epsilon}} - I = -\begin{pmatrix} \frac{1}{D_J}\frac{s_1}{s_0}-1 & \dots & \frac{1}{D_J}\frac{s_J}{s_0} \\ & \ddots & \\ \frac{1}{D_J}\frac{s_1}{s_0} & \dots & \frac{1}{D_J}\frac{s_J}{s_0}-1 \end{pmatrix}.$$

Thus, $\det\left(D_p\hat{\boldsymbol{\epsilon}}-I\right) = (-1)^J\left(\frac{1}{D_J s_0} - \frac{1-D_J}{D_J}\right)$.

Next, note that for a for a any $f\in\mathscr{F}$, if we define $s_0^f(\mathbf{p}) = 1-\sum_{j\in\mathscr{J}_f} s_j(\mathbf{p})$, then

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\sigma}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \dots & s_1 s_{J_f} \\ s_2 s_1 & \ddots & & s_2 s_{J_f} \\ \vdots & & \ddots & \vdots \\ s_{J_f}s_1 & \dots & s_{J_f}s_{J_f-1} & -s_{J_f}(1-s_{J_f}) \end{pmatrix}^{-1} = -\frac{\sigma}{\alpha s_0^f}\begin{pmatrix} 1+\frac{s_0^f}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+\frac{s_0^f}{s_1} \end{pmatrix}..$$

Further, $\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega} = \sum_j s_j \omega_j^f$ and

$$D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}) = \frac{\alpha}{\sigma}\begin{pmatrix} s_1\big(\sum_{j\in\mathscr{J}} s_j\omega_j^f - \omega_1^f\big) \\ \vdots \\ s_{J_f}\big(\sum_{j\in\mathscr{J}} s_j\omega_j^f - \omega_{J_f}^f\big) \end{pmatrix}$$

Hence,

$$\begin{aligned}(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}) &= -\frac{1}{s_0^f}\begin{pmatrix} \big(\sum_{j\in\mathscr{J}_f} s_j + s_0^f\big)\sum_{j\in\mathscr{J}} s_j\omega_j^f - \sum_{j\in\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_1^f \\ \vdots \\ \big(\sum_{j\in\mathscr{J}_f} s_j + s_0^f\big)\sum_{j\in\mathscr{J}} s_j\omega_j^f - \sum_{j\in\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_{J_f}^f \end{pmatrix} \\ &= -\frac{1}{s_0^f}\begin{pmatrix} \sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_1^f \\ \vdots \\ \sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_{J_f}^f \end{pmatrix}\end{aligned}$$

Further, since

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}) = = \frac{\sigma}{\alpha s_0^f}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Finally,

$$g^f(\mathbf{p}) = -\begin{pmatrix} \frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_1^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha} \\ \vdots \\ \frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_{J_f}^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha} \end{pmatrix}$$

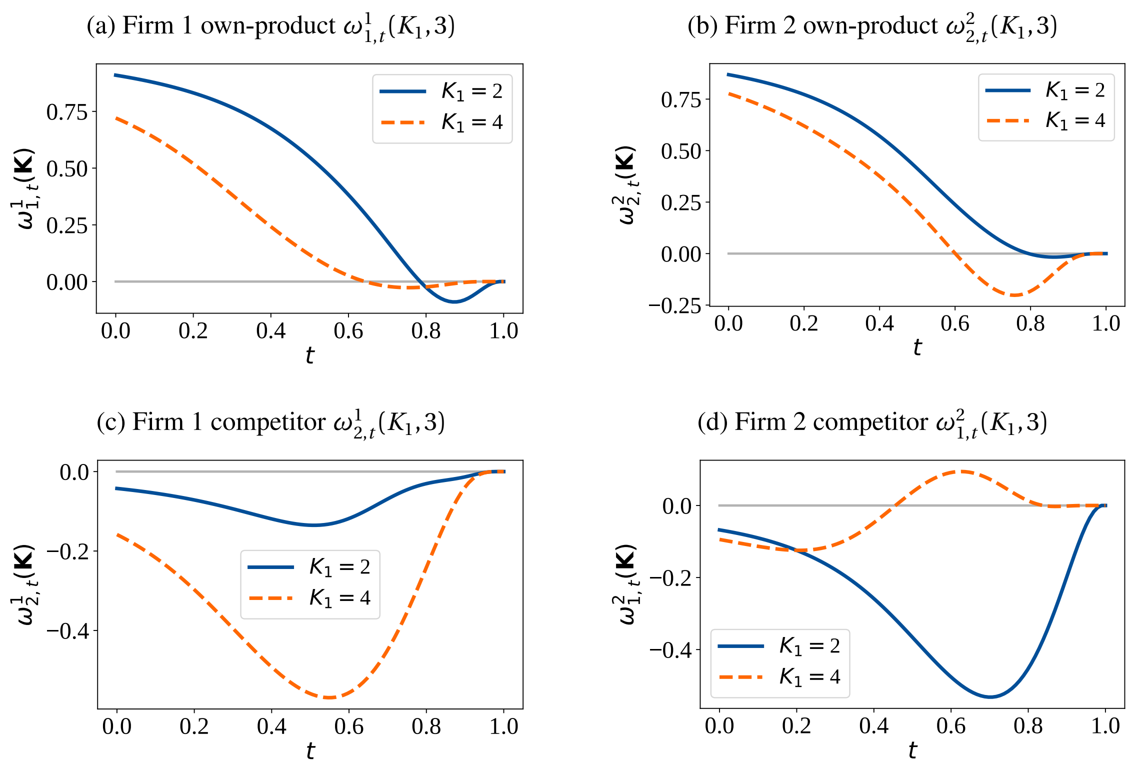
and

$$D_{\mathbf{p}} g^f(\mathbf{p}) = \left(\begin{pmatrix} \frac{s_k}{1-\sum\limits_{j\in\mathcal{J}_f} s_j} \\ \vdots \\ \frac{s_k}{1-\sum\limits_{j\in\mathcal{J}_f} s_j} \end{pmatrix}_{k\in\mathcal{J}_f}, \begin{pmatrix} \frac{\alpha}{\sigma} \frac{-(1-\sum\limits_{j\in\mathcal{J}_f} s_j)(1-\sum\limits_{j\notin\mathcal{J}_f} s_j)+(\sum\limits_{j\in\mathcal{J}_f} s_j)(\sum\limits_{j\notin\mathcal{J}_f} s_j)}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \omega_j^f + s_k \frac{\sum\limits_{j\in\mathcal{J}_f} s_j}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \\ \vdots \\ \frac{\alpha}{\sigma} \frac{-(1-\sum\limits_{j\in\mathcal{J}_f} s_j)(1-\sum\limits_{j\notin\mathcal{J}_f} s_j)+(\sum\limits_{j\in\mathcal{J}_f} s_j)(\sum\limits_{j\notin\mathcal{J}_f} s_j)}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \omega_j^f + s_k \frac{\sum\limits_{j\in\mathcal{J}_f} s_j}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \end{pmatrix}_{k\notin\mathcal{J}_f} \right)$$

For large σ the term in front of ω_j^f vanishes relative to the probability.

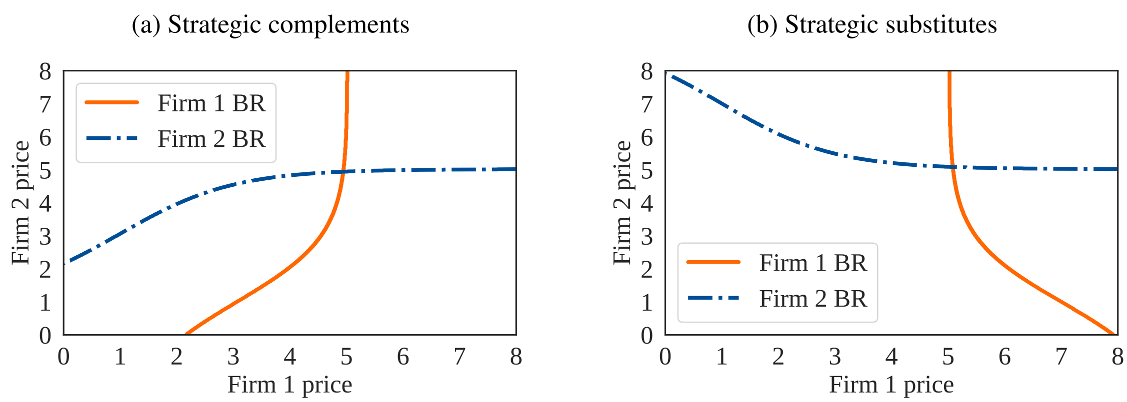
D Additional Empirical Results

Figure 11: Simulated scarcity effects for $K_2 = 3$, K_1 varying



Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 12: Strategic complements and substitutes in the stage game

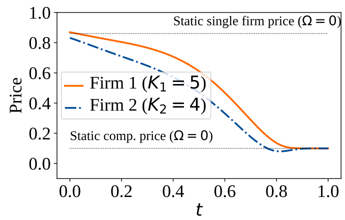


Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$, as well as $\omega_1^1 = \omega_2^2 = 4$. Panel (a) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = 4$. Panel (b) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = -4$.

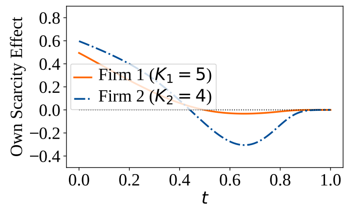
Figure 13: Simulated prices and scarcity effects

$\mathbf{K} = (5, 4)$

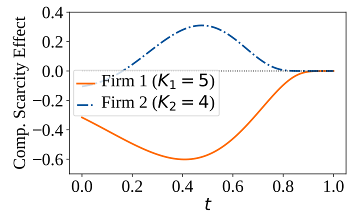
(a) Price paths over time



(b) Own ω over time

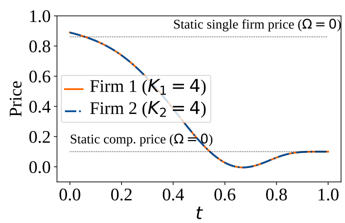


(c) Competitor ω over time

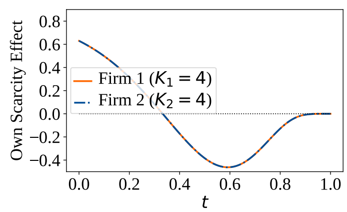


$\mathbf{K} = (4, 4)$

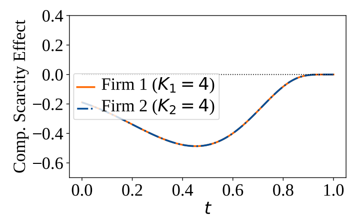
(d) Price paths over time



(e) Own ω over time

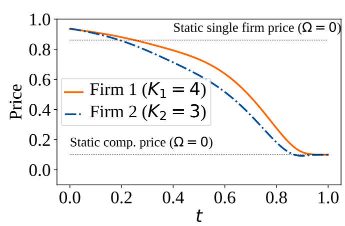


(f) Competitor ω over time

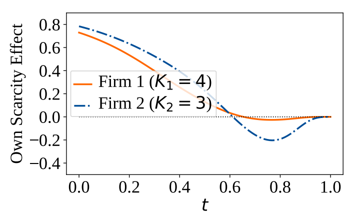


$\mathbf{K} = (3, 4)$

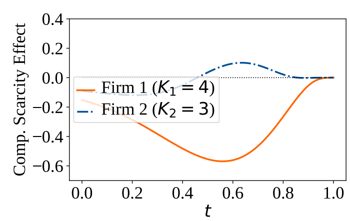
(g) Price paths over time



(h) Own ω over time



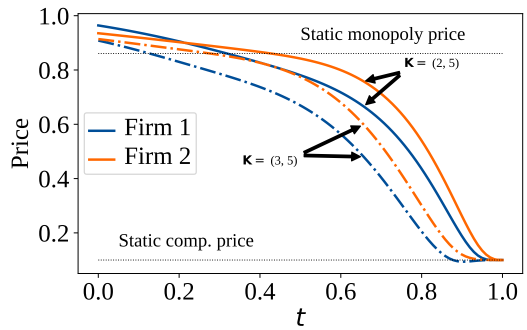
(i) Competitor ω over time



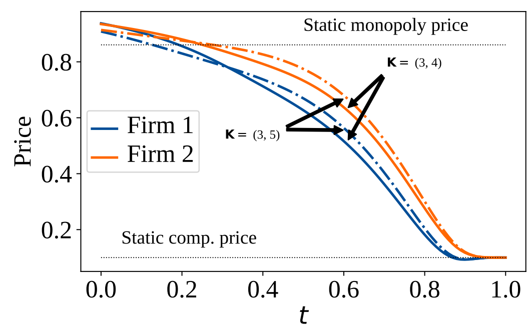
Notes: The simulations assume $\delta = (1, 1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 14: Price paths for varying levels of capacity

(a) Sale of a product with minimum inventory



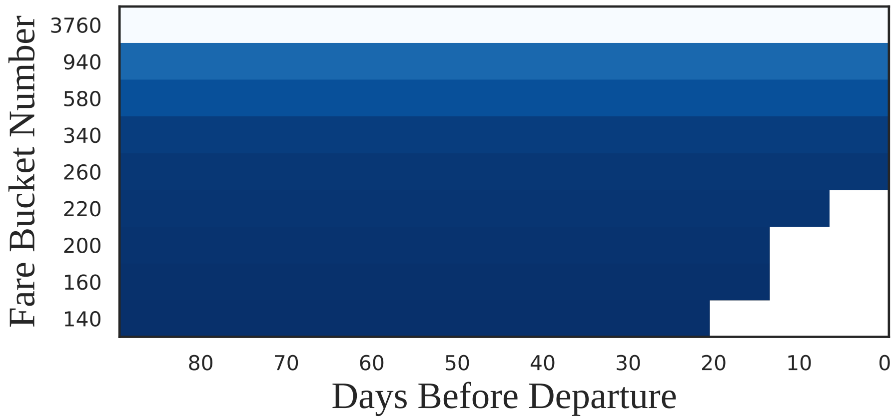
(b) Sale of a product without minimum inventory



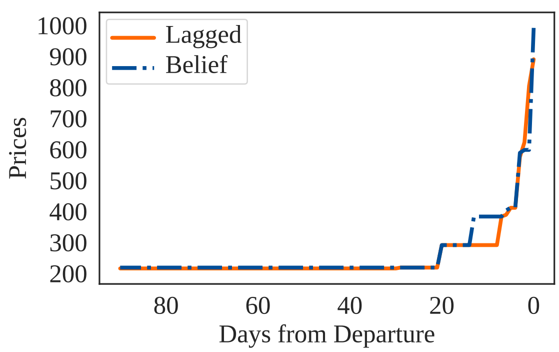
Notes: These simulations correspond to logit demand with parameter values $\delta_j = 1$, $\alpha = 1$, $\lambda = 10$ and scale factor $\rho = 0.05$. Panel (a) shows both firm's price paths for $\mathbf{K} = (3,5)$ and $\mathbf{K} = (2,5)$. Panel (b) shows both firm's price paths for $\mathbf{K} = (3,5)$ and $\mathbf{K} = (3,4)$.

Figure 15: Heuristic Models Pricing Example

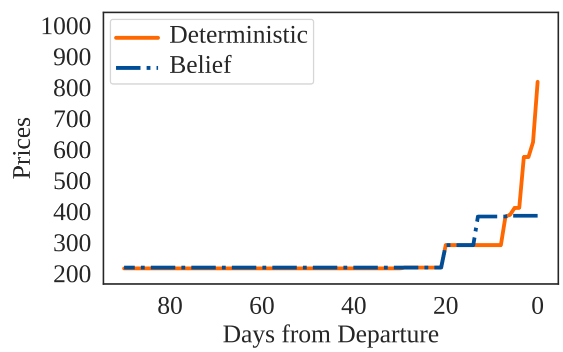
(a) Deterministic Model



(b) Lagged-Price Model



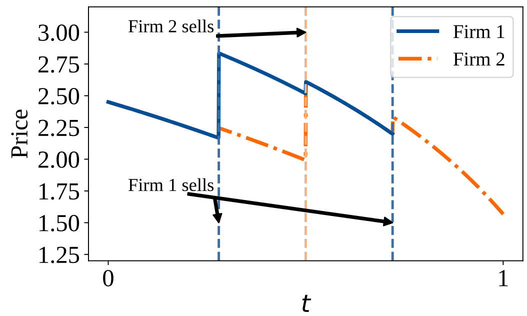
(c) Deterministic Model



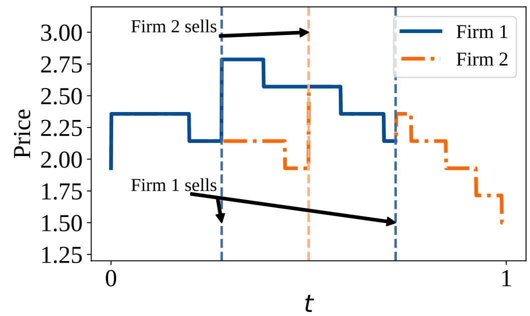
Note: Panel (a) shows a firm's fares and belief of the other firm's price over time for an instance of the simulation in the lagged-price model. Panel (b) shows a firm's fares and belief of the other firm's price over time for an instance of the simulation in the deterministic model.

Figure 16: Price Path Realizations comparing Benchmark model to Heuristics

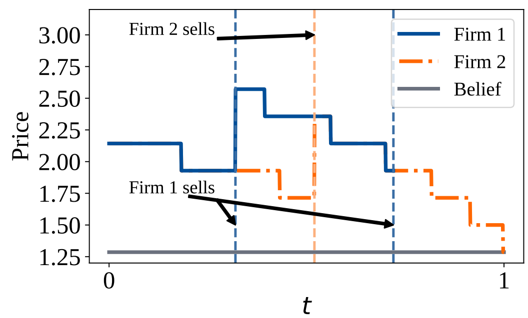
(a) Price paths for the benchmark model



(b) Price paths for the lagged algorithm



(c) Price paths for the deterministic algorithm



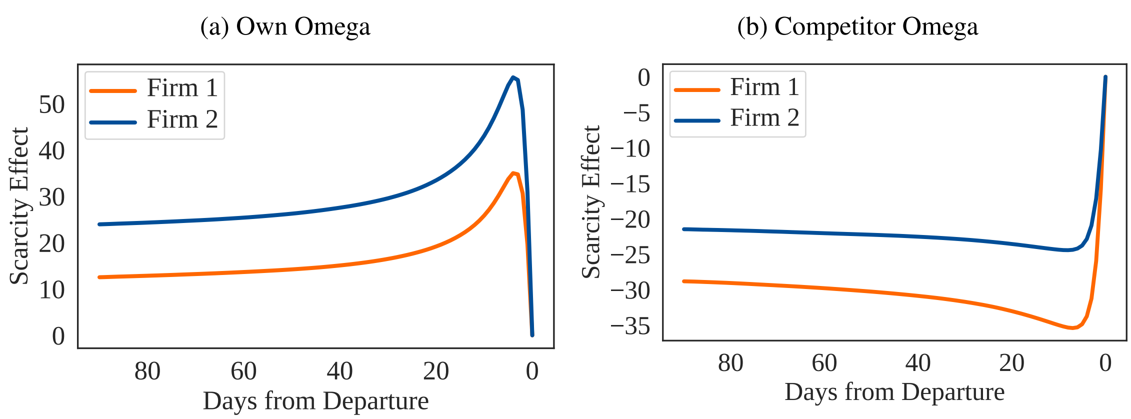
Notes: We assume demand follows a logit specification with an initial capacity vector of $\mathbf{K}_0 = (2,2)$. Time is continuous for $t \in [0,1]$. There are three panels: panel (a) depicts the equilibrium price path for the benchmark model, panel (b) considers prices if firms use the lagged model, and panel (c) considers prices if firms use the deterministic model. The vertical lines mark realized sales times; the color denotes the firm that received the sale. These simulations correspond to the parameter values $\delta_j = 1$, $\alpha = 1$, $\rho = 1$, $\lambda = 10$ and $\mathbf{K}_0 = [2,2]$. In the heuristic model, firms assume that the competitor prices at the level given by the grey line.

Table 4: Recreation of Table 3 with restricted initial capacity

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Deterministic	94.1	95.5	95.9	108.4	103.0	105.1	102.0	178.3
Lagged	102.0	100.3	101.2	104.4	102.9	100.6	100.2	104.0
Uniform	97.5	78.1	77.3	113.7	98.5	101.1	99.9	242.0

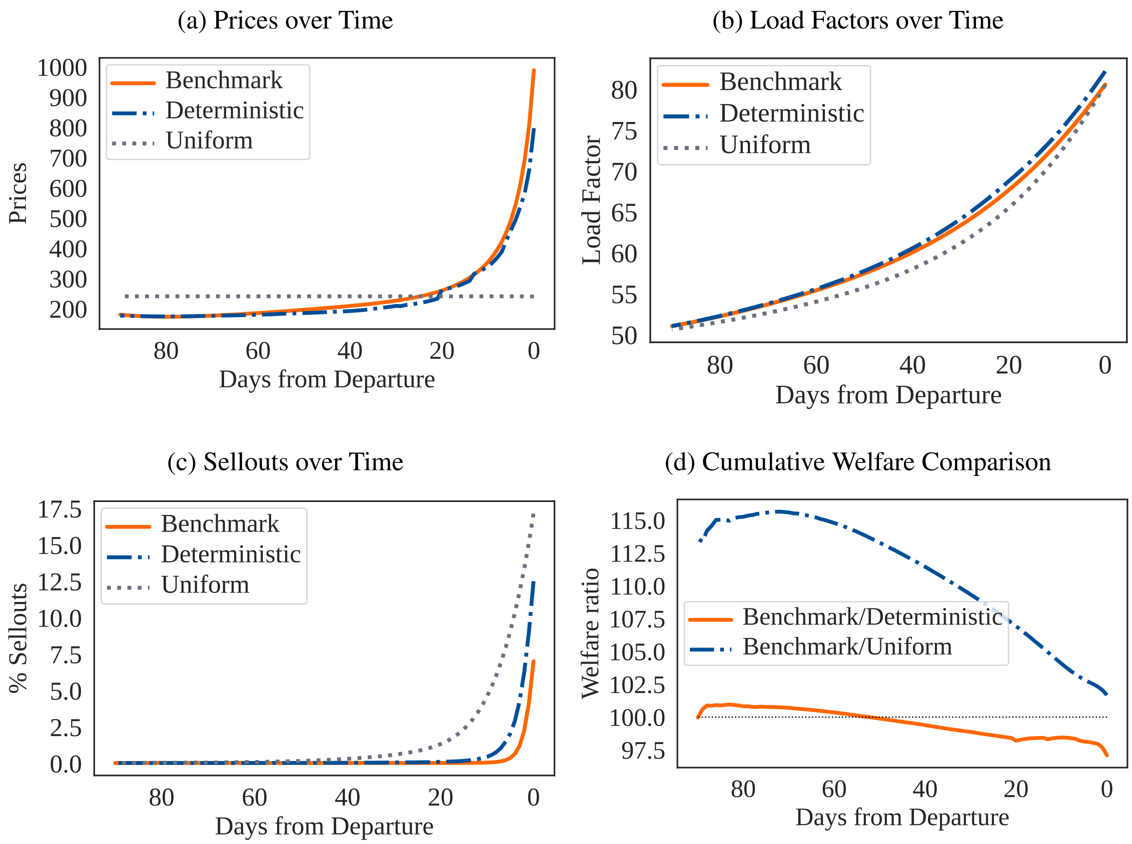
Note:

Figure 17: Recreation of Fig. 7 with restricted initial capacity



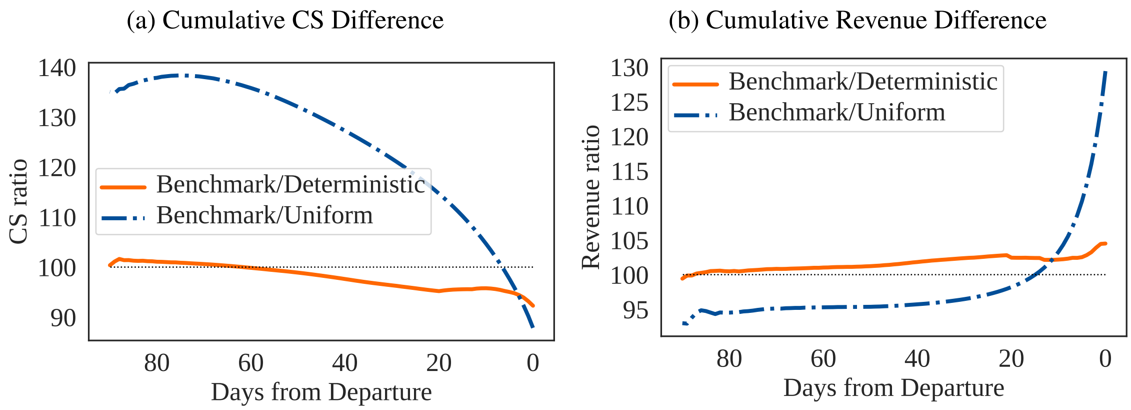
Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Figure 18: Recreation of Fig. 8 with restricted initial capacity



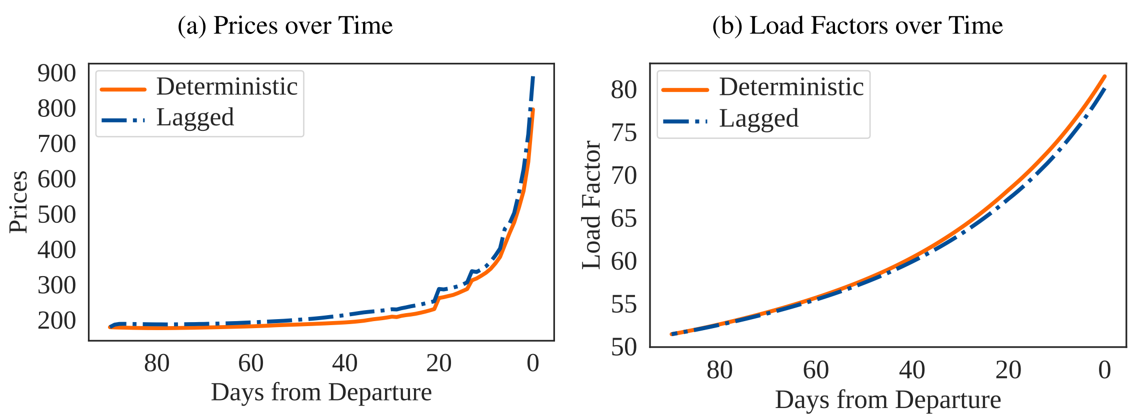
Note: Panel (a) shows the average price over time for the benchmark, deterministic and uniform models. Panel (b) shows the average load factors over time for the same three models. Panel (c) shows the average sellouts over time for the same three models. Panel (d) shows the ratio of average cumulative welfare for the benchmark model with respect to the deterministic one, and for the benchmark model with respect to the uniform one.

Figure 19: Recreation of Fig. 9 with restricted initial capacity



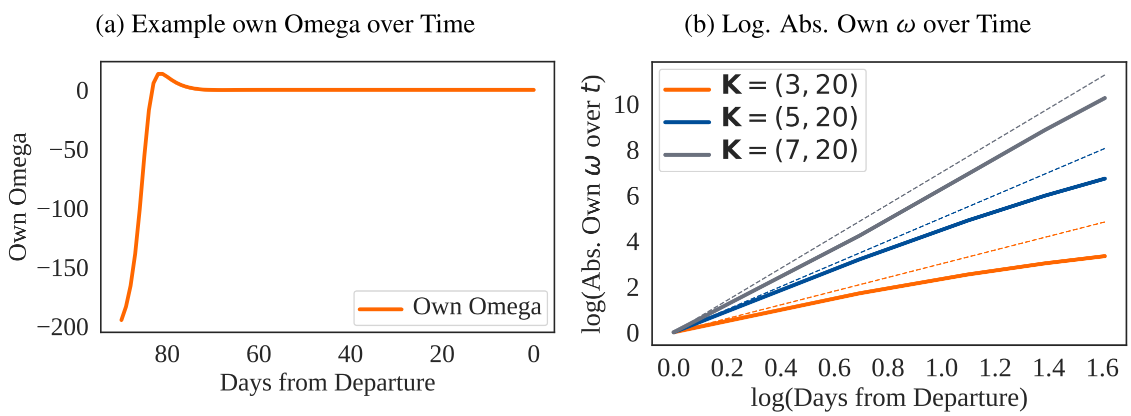
Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Figure 20: Recreation of Fig. 10 with restricted initial capacity



Note: Panel (a) shows the average prices over time for the two heuristic models. Panel (b) shows the average load factors over time for the same two models.

Figure 21: Example of a negative own Opportunity Costs



Note: Panel (a) shows the own ω over time for a given state in one of our Benchmark solutions. Panel (b) shows the log of the absolute value of the own ω over time for three states in one of our Benchmark solutions. The dotted lines represent the behavior these curves would follow if the omegas were proportional to $|T-t|^{\min(\mathbf{K})}$.