Robust Bounds for Welfare Analysis

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Motivation

- Many papers in economics have the following structure:

  1. A policy (*e.g.*, tax/subsidy) was implemented.
  2. Using prices and quantities before and after, estimate demand.
  3. Impute the change in welfare + compare to costs/revenues.

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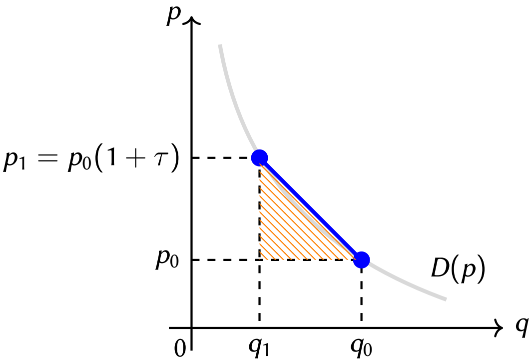
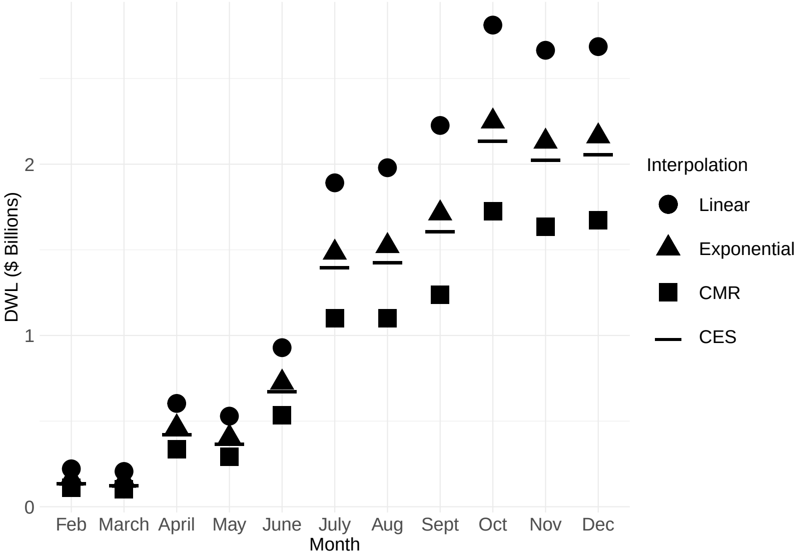
  → Functional forms (*e.g.*, CES or linear demand) are often assumed for convenience.

Example: evaluating the deadweight loss of the Trump tariffs



- Amiti, Redding and Weinstein (2019)
- **Setting:** 2018 trade war involved tariffs as high as 30–50%.
- **Question:** What was the DWL?
- **Approach:** Compare monthly prices & quantities by item in 2017 vs. 2018.
- **Method:** Approximate $D(p)$ with a linear curve; integrate under the curve.

Bounding the DWL across countries and products



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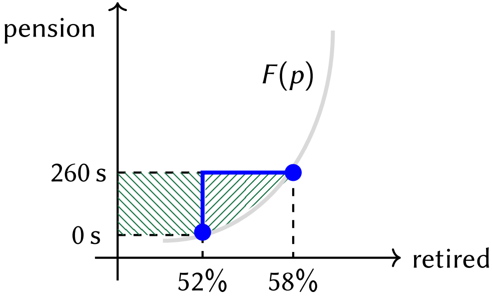
  → Conservative bounds in lieu of assumptions are often extreme.

Example: WTP of 1911 UK pension recipients



- Giesecke and Jäger (2021)
- **Setting:** Pensions created for poor 70+ year olds in 1911.
- **Question:** What is the MVPF of the pension policy?
- **Approach:** MVPF = (WTP for not working) / (cost of pension).
- **Method:** Compute % marginal workers via RD; assume marginal workers' WTP = 0.

Motivation

- Many papers in economics have the following structure:
  1. A policy (*e.g.*, tax/subsidy) was implemented.
  2. Using prices and quantities before and after, estimate demand.
  3. Impute the change in welfare $+$ compare to costs/revenues.
- Measuring welfare requires taking a stance on what the demand curve looks like at unobserved points.
  - Functional forms (*e.g.*, CES or linear demand) are often assumed for convenience.
  - Conservative bounds in lieu of assumptions are often extreme.
  - $\rightsquigarrow$ Is there a more principled way to engage with assumptions and evaluate welfare?

This paper

- Instead of interpolating to get a welfare estimate, we establish **welfare bounds**.
  - These bounds are **robust**: they give the *best-case* and *worst-case* welfare estimates that are consistent with a set of pre-specified economic assumptions.
  - These bounds are also **simple**: we can compute them in closed form.

This is a tool for empirical microeconomists

- Our bounds apply directly to settings with:
  - (i) exogenous policy shocks/experiments/quasi-experiments;
  - (ii) measurements of "price" and "quantity," before and after the policy shock; and
  - (iii) interest in effects on consumer surplus (or other welfare measures).

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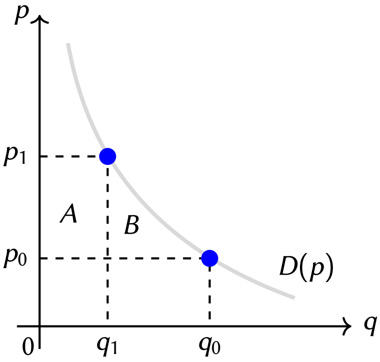
  - (iii) interest in effects on consumer surplus (or other welfare measures).

- We show how our bounds can be applied to a variety of settings across literatures:
  - #1. deadweight loss of import tariffs (Amiti, Redding and Weinstein, 2019)
  - #2. welfare impact of energy subsidies (Hahn and Metcalfe, 2021)
  - #3. willingness to pay for the Old-Age Pension Act (Giesecke and Jäger, 2021)
  - #4. marginal excess burden of income taxation (Feldstein, 1999)

An analyst observes 2 points on a demand curve: (p_0, q_0) and (p_1, q_1).

Question. What is the change in consumer surplus from (p_0, q_0) to (p_1, q_1)?



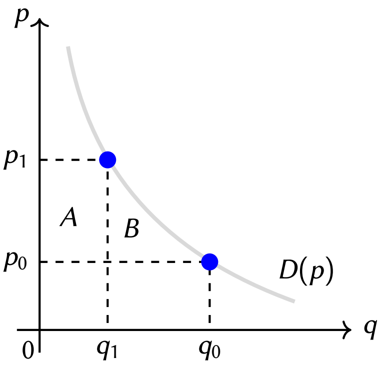
- **Main challenge:** $D(p)$ isn't observed.
- With $D(p)$, change in CS is equal to

$$\underbrace{\text{area } A}_{=(p_1-p_0)q_1} + \text{area } B = \int_{p_0}^{p_1} D(p)\, \mathrm{d}p.$$

- Equivalently, we want to *bound* area B.

Bounds without additional assumptions

- Using only the fact that the demand curve is decreasing, the analyst can establish bounds on the change in welfare (Fogel, 1964; Varian, 1985).



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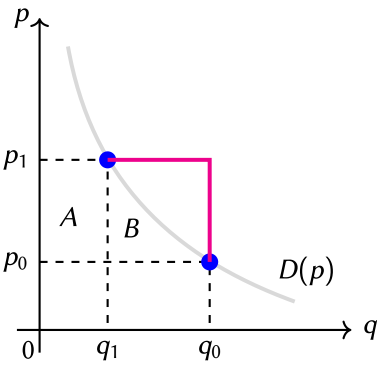

- An upper bound on area B is

$$\text{area } B \leq (p_1 - p_0) \times (q_0 - q_1) .$$

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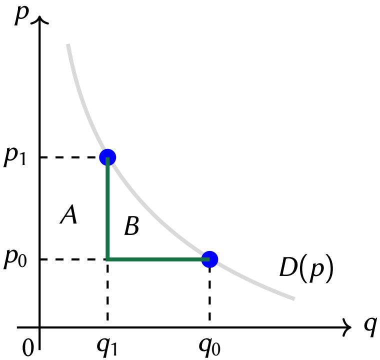

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$$0 \leq \text{area } B.$$

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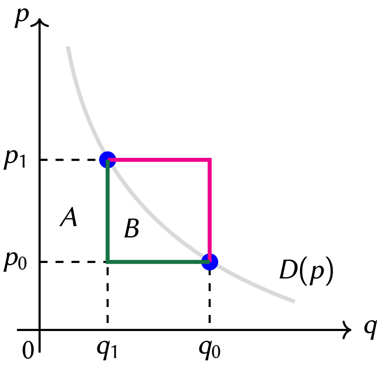

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$$\text{area } B \leq (p_1 - p_0) \times (q_0 - q_1) \,.$$

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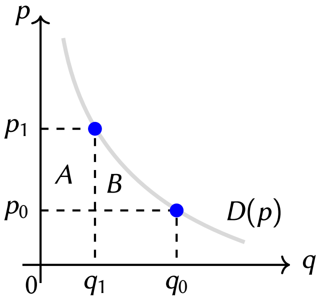
- These bounds are attained only when **elasticities are equal to 0 or $-\infty$**.

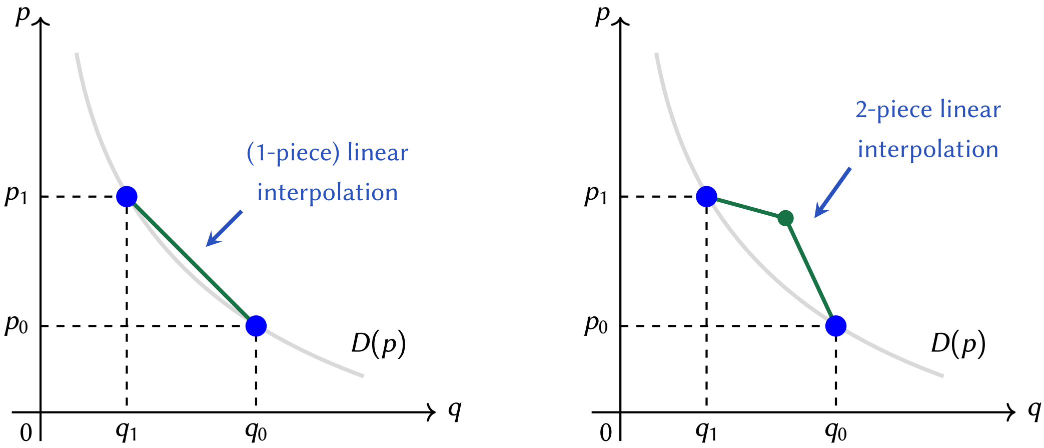
An analyst observes 2 points on a demand curve: (p_0, q_0) and (p_1, q_1).

We assume that elasticities between (p_0, q_0) and (p_1, q_1) lie in the interval $[\underline{\varepsilon}, \overline{\varepsilon}] \subset \mathbb{R}_{\leq 0}$.

Question. What is the change in consumer surplus from (p_0, q_0) to (p_1, q_1)?

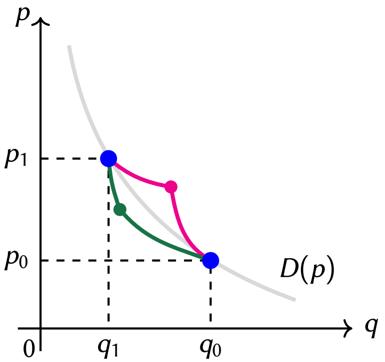


Defining 1-piece and 2-piece interpolations



Theorem 1 (welfare bounds).

The upper and lower bounds for the change in consumer surplus are attained by **2-piece CES interpolations**. ▸ Give proof ▸ Skip proof



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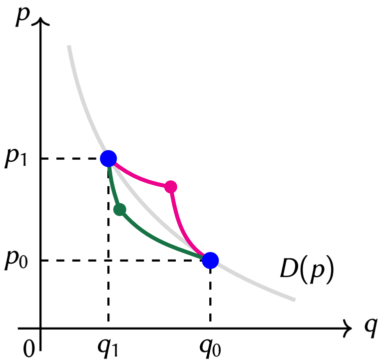
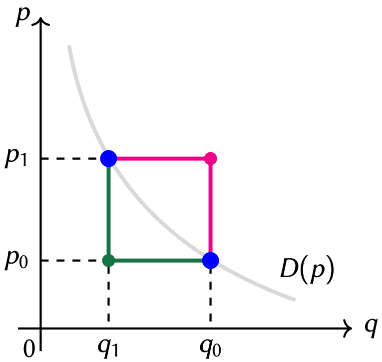
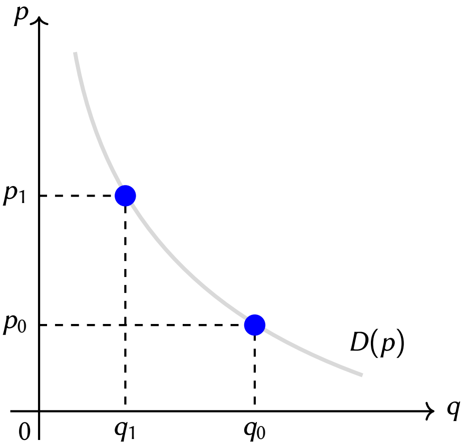
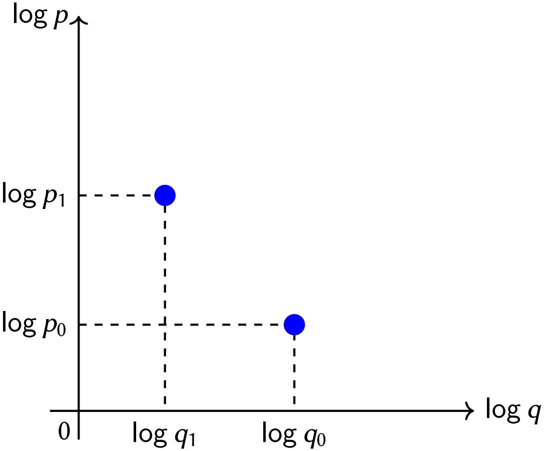


$$\overline{\varepsilon} \to 0,$$
$$\underline{\varepsilon} \to -\infty$$



Geometric derivation of welfare bounds ◂ Back





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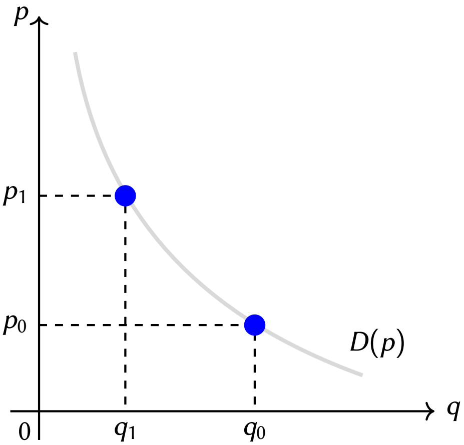

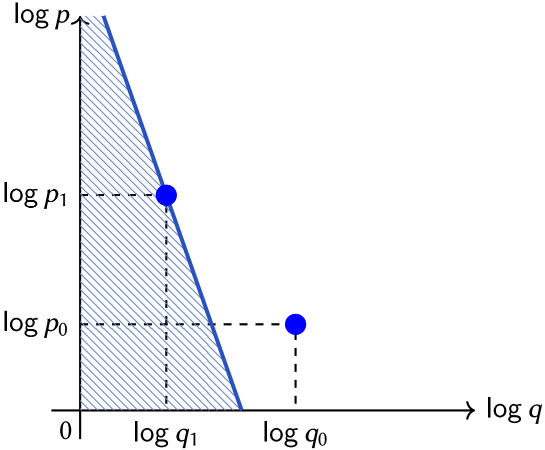

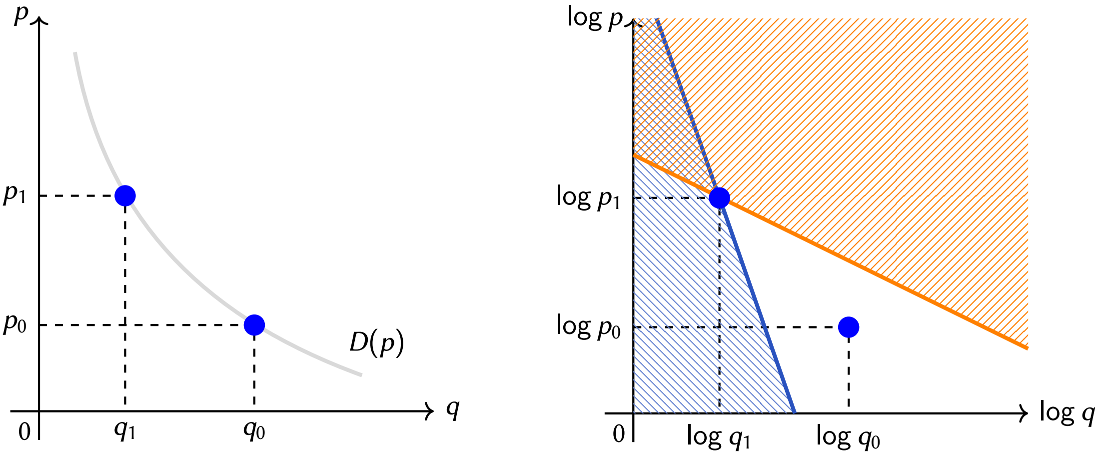

p
p1
p0
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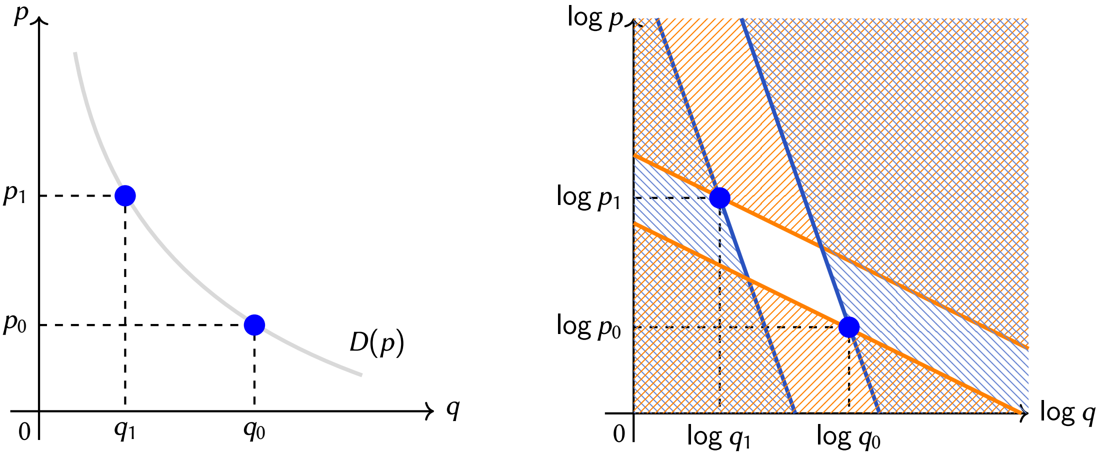
log q1
log q0
log q

Geometric derivation of welfare bounds



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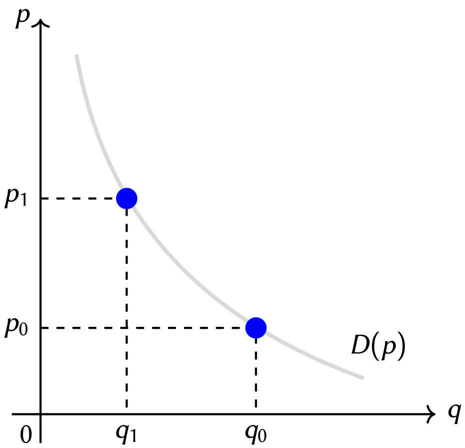

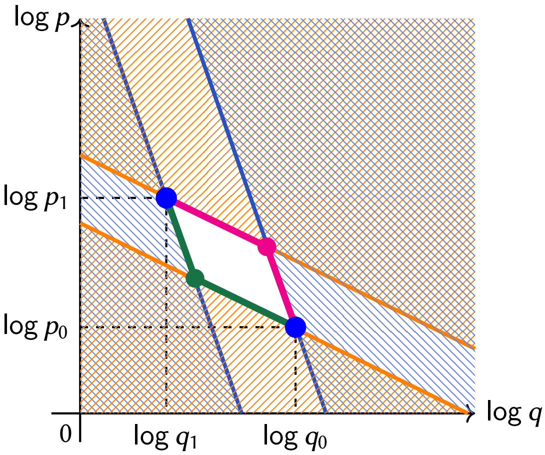

Geometric derivation of welfare bounds
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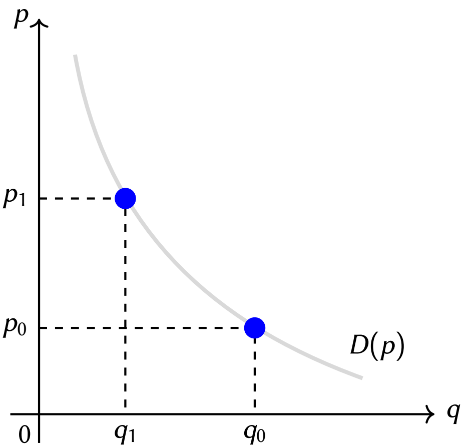

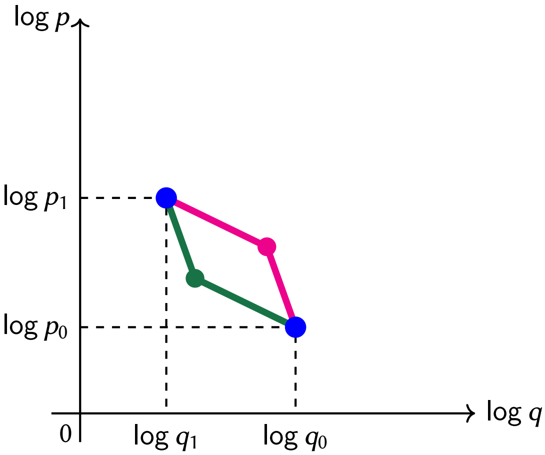

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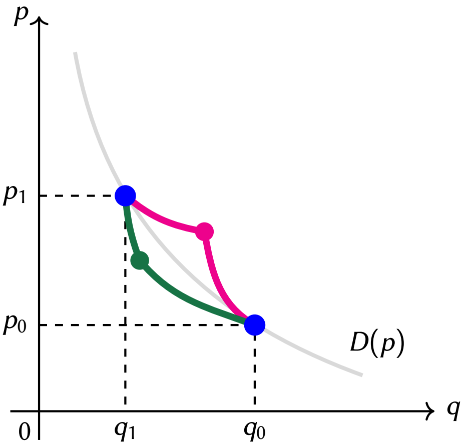

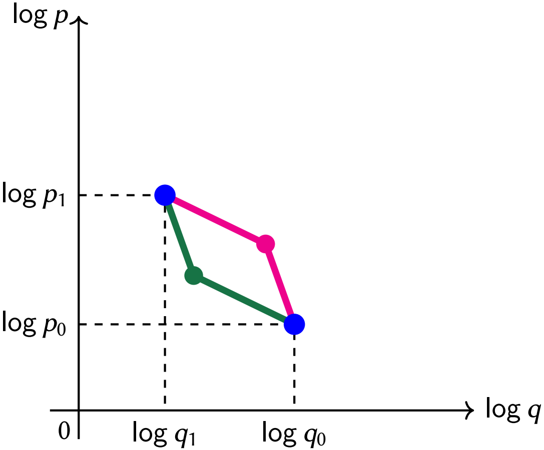

Choosing elasticity bands

- **Question.** What is a reasonable elasticity band?

  (a) Combine estimates from the literature.

  $\rightsquigarrow$ *E.g., "estimates of short run gasoline elasticities are between* -0.2 *and* -0.4*."*

  (b) Draw upon institutional knowledge.

  $\rightsquigarrow$ *E.g., "at the extreme, elasticities can't possibly be lower than* -5*."*

  (c) Draw a (symmetric) band around the *average* elasticity.

$$\underline{\varepsilon} \leq \frac{\log q_1 - \log q_0}{\log p_1 - \log p_0} \leq \overline{\varepsilon}.$$

Discussion of basic model

Our welfare bounds for the basic model rely on a number of modeling choices:

1. No assumption is made about the curvature of the demand curve.

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3. Only two points (p_0, q_0) and (p_1, q_1) on the demand curve are observed.

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4. The points (p_0, q_0) and (p_1, q_1) on the demand curve are observed precisely.

   In practice, the analyst might be limited by sampling error.

Extensions to basic model

Our welfare bounds for the basic model rely on a number of modeling choices:

1. *In practice, the analyst might make assumptions about demand curvature.*

   $\Longrightarrow$ We show how **demand curvature** assumptions lead to tighter bounds.

2. *In practice (e.g., counterfactuals), the analyst might observe p_0, p_1, and q_1, but not q_0.*

   $\Longrightarrow$ We show how to **extrapolate** from fewer observations.

3. *In practice, the analyst might observe more points on the demand curve.*

   $\Longrightarrow$ We show how to **interpolate** with more observations.

4. *In practice, the analyst might be limited by sampling error.*

   $\Longrightarrow$ We show how to incorporate **sampling error** into welfare bounds.

① Assumptions on demand curvature

"Notice that **these results depend on the fact** that the *PP* curve slopes upward, which in turn depends on the assumption that the **elasticity of demand falls with** c.

This assumption, which might alternatively be stated as an assumption that the elasticity of demand rises when the price of a good is increased, **seems plausible**.

In any case, it seems to be **necessary** if this model is to yield reasonable results, and I make the assumption without apology."

—Krugman (1979)

① Assumptions on demand curvature

Many models across different fields impose additional assumptions on demand:

(A1) Decreasing elasticity, or "Marshall's second law." (Marshall, 1890; Krugman, 1979)

(A2) Decreasing marginal revenue. (Myerson, 1981; Bulow and Roberts, 1989)

(A3) Log-concave demand. (Caplin and Nalebuff, 1991a; Bagnoli and Bergstrom, 2005)

(A4) Concave demand. (Rosen, 1965; Szidarovszky and Yakowitz, 1977; Caplin and Nalebuff, 1991a)

(A5) ρ-concave demand that generalizes **(A3)** and **(A4)**. (Caplin and Nalebuff, 1991a,b)

We call these "**concave-like** assumptions" on demand.

① Assumptions on demand curvature

Many models across different fields impose additional assumptions on demand:

(A6) Convex demand. (Svizzero, 1997; Aguirre, Cowan and Vickers, 2010; Tsitsiklis and Xu, 2014)

(A7) Log-convex demand. (Caplin and Nalebuff, 1991b; Aguirre, Cowan and Vickers, 2010)

(A8) ρ-convex demand that generalizes **(A6)** and **(A7)**. (Caplin and Nalebuff, 1991a,b)

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Relationships between curvature assumptions

Concave-like assumptions

(A1) Decreasing elasticity

(A2) Decreasing MR

(A3) Log-concave demand

(A4) Concave demand

(A5) ρ-concave demand

$$(A4) \Longrightarrow (A3) \Longrightarrow (A1), \qquad (A3) \Longrightarrow (A2)$$

Convex-like assumptions

(A6) Convex demand

(A7) Log-convex demand

(A8) ρ-convex demand

$$(A7) \Longrightarrow (A6).$$

Theorem 2a. (concave-like assumptions).

The **lower** bound for the change in consumer surplus are attained by:

(A1) **decreasing elasticity:** a *CES* interpolation; $D(p) = \theta_1 p^{-\theta_2}$

(A2) **decreasing MR:** a *constant MR* interpolation; $D(p) = \theta_1 (p - \theta_2)^{-1}$

(A3) **log-concave demand:** an *exponential* interpolation; $D(p) = \theta_1 e^{-\theta_2 p}$

(A4) **concave demand:** a *linear* interpolation; $D(p) = \theta_1 - \theta_2 p$

(A5) **ρ-concave demand:** a *ρ-linear* interpolation. $D(p) = [1 + \rho (\theta_1 - \theta_2 p)]^{1/\rho}$

Theorem 2b. (convex-like assumptions).

The **upper** bound for the change in consumer surplus are attained by:

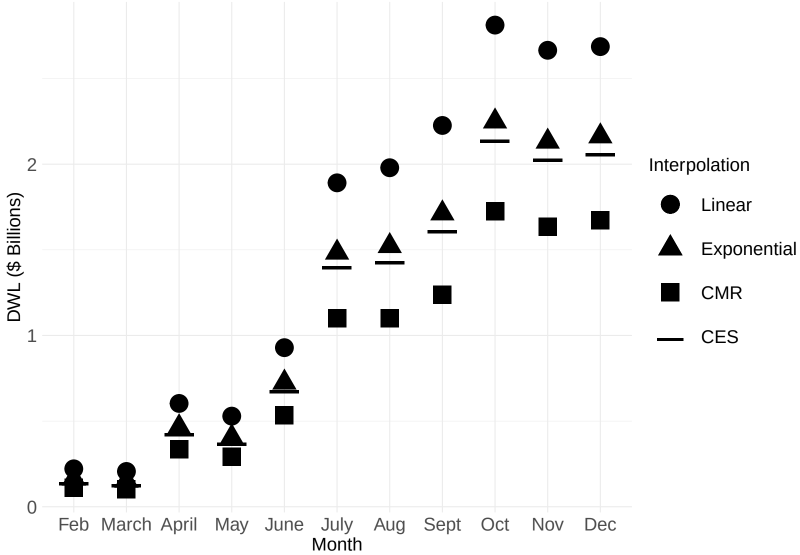
(A6) convex demand: a *linear* interpolation; $D(p) = \theta_1 - \theta_2 p$

(A7) log-convex demand: an *exponential* interpolation; $D(p) = \theta_1 e^{-\theta_2 p}$

(A8) ρ-convex demand: a ρ-*linear* interpolation. $D(p) = [1 + \rho(\theta_1 - \theta_2 p)]^{1/\rho}$

▸ Geometric intuition

Bounding the tariff DWL across countries and products



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   $\implies$ We show how **demand curvature** assumptions lead to tighter bounds.

2. *In practice (e.g., counterfactuals), the analyst might observe p_0, p_1, and q_1, but not q_0.*

   $\implies$ We show how to **extrapolate** from fewer observations.

3. *In practice, the analyst might observe more points on the demand curve.*

   $\implies$ We show how to **interpolate** with more observations.

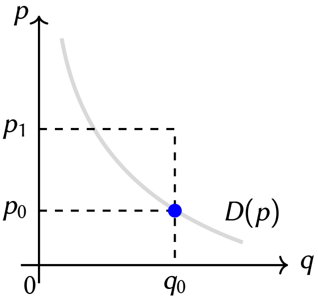
4. *In practice, the analyst might be limited by sampling error.*

   $\implies$ We show how to incorporate **sampling error** into welfare bounds.

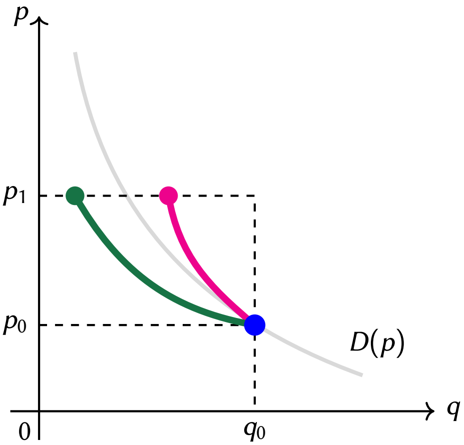
An analyst observes **1 point** on a demand curve: (p_0, q_0); p_1 is given.

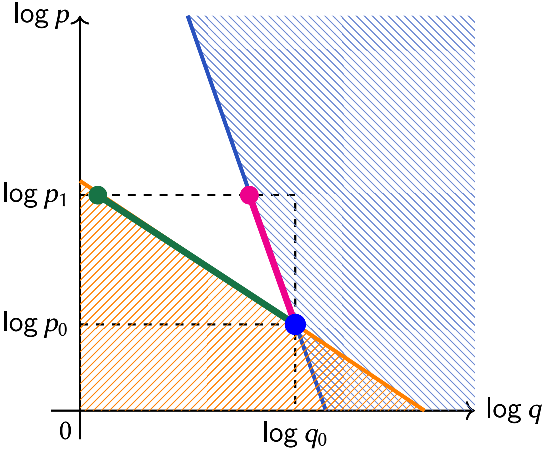
We assume that elasticities between p_0 and p_1 lie in the interval $[\underline{\varepsilon}, \overline{\varepsilon}] \subset \mathbb{R}_{\leq 0}$.

Question. What is the change in consumer surplus from p_0 to p_1?

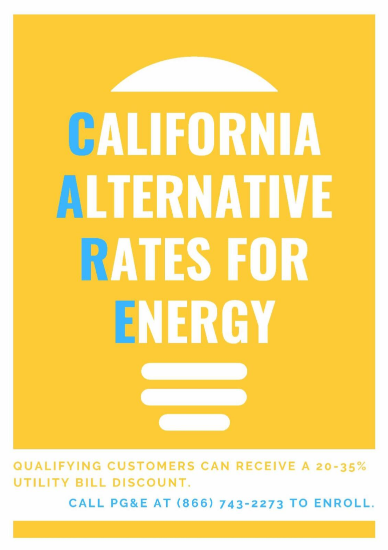


② Extrapolating from less data: geometric intuition





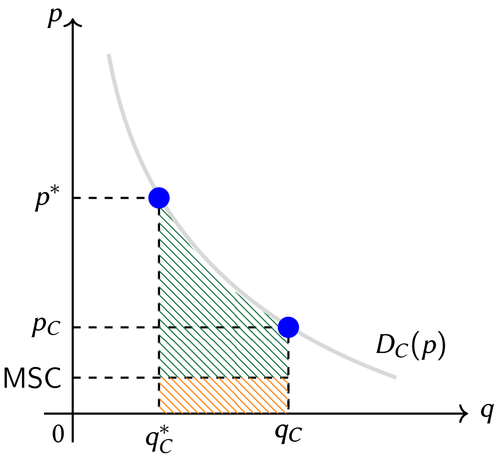
What is the welfare impact of CARE gas subsidies?

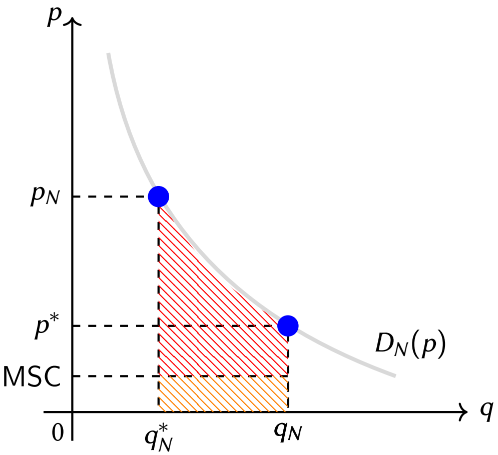


CARE Program:

- **Low income:** 20% discount on gas
  - ⇝ Gas usage ↑
  - ⇝ Consumer surplus ↑
  - ⇝ Climate impact ↓
- **Other households:** Gas price ↑ (given a fixed budget)
  - ⇝ Gas usage ↓
  - ⇝ Consumer surplus ↓
  - ⇝ Climate impact ↑
- **Administrative Cost:** $7M

Bounding counterfactual welfare from uniform pricing





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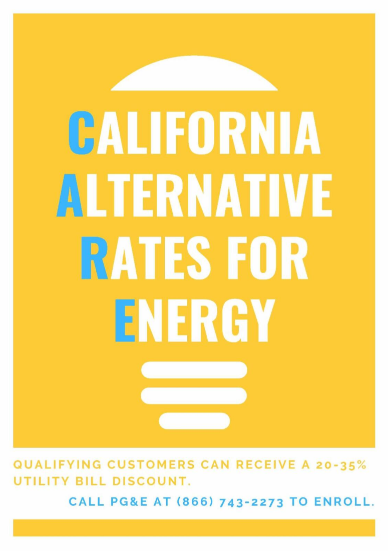

CARE Program:

- **Low income:** 20% discount on gas
  - $\rightsquigarrow$ Gas usage $\uparrow$
  - $\rightsquigarrow$ Consumer surplus $\uparrow$
  - $\rightsquigarrow$ Climate impact $\downarrow$
- **Other households:** Gas price $\uparrow$ (given a fixed budget)
  - $\rightsquigarrow$ Gas usage $\downarrow$
  - $\rightsquigarrow$ Consumer surplus $\downarrow$
  - $\rightsquigarrow$ Climate impact $\uparrow$
- **Administrative Cost:** $7M

Question: Is CARE net welfare improving?

Welfare impact of energy subsidies (Hahn and Metcalfe, 2021)

- **Empirical strategy:**
  - Randomly nudge eligible households to sign up for CARE.
  - Compute LATE based on gas usage with and without CARE (using nudges as an IV).
  - Interpret the LATE as an elasticity:

  ⇝ *How much does gas usage change given a 20% discount in unit price?*

Welfare impact of energy subsidies (Hahn and Metcalfe, 2021)

- **Empirical strategy:**
  - Randomly nudge eligible households to sign up for CARE.
  - Compute LATE based on gas usage with and without CARE (using nudges as an IV).
  - Interpret the LATE as an elasticity:
  - ⇝ *How much does gas usage change given a 20% discount in unit price?*
- **Modeling assumptions:**
  - The CARE program operates under a fixed budget
  - ⇝ The **counterfactual** "uniform" **price** is pinned down by observed quantities

$$N_n \left(P_n - P^*\right) Q_n = N_c \left(P^* - P_c\right) Q_c + A.$$

  - Consumer demand is linear

Welfare impact of energy subsidies (Hahn and Metcalfe, 2021)

- **Elasticity estimates:**
  - ⇝ Estimated CARE elasticity of $-\mathbf{0.35}$.
  - – Assume non-CARE elasticity is $-\mathbf{0.14}$ (Auffhammer and Rubin, 2018).
- **Welfare estimates:**

CARE:	$+\ \$5.3M$
Non-CARE:	$-\ \$3.1M$
Admin Costs:	$-\ \$7.0M$
Net:	$-\ \$4.8M$

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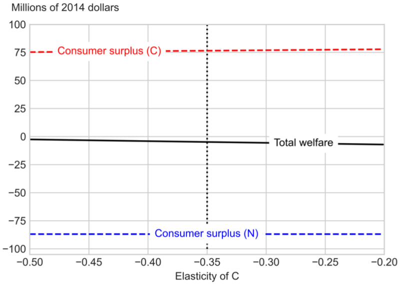
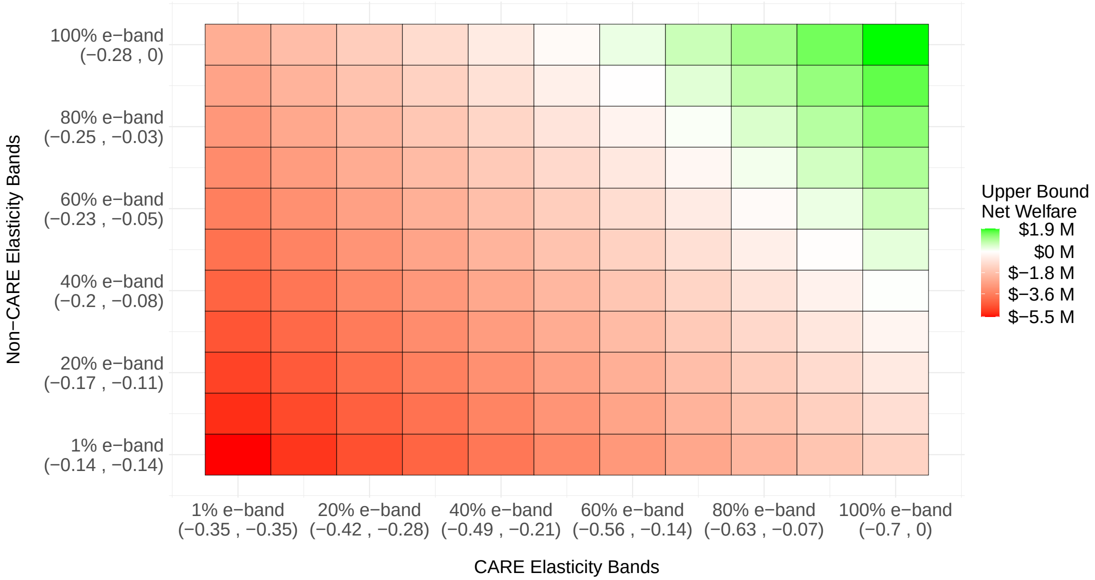


How robust is the negative welfare result?



Discussion

- **Why might we expect the welfare results to flip?**
  - **#1.** Before imposing any assumptions, we can test the conservative (box) bounds.
  - **#2.** We "observe" p_1, q_1, ε_1 and p_0 but not q_0 or ε_0.
  - **#3.** Our bounds are "adversarial."

- **So, how do we interpret these results?**
  - ⇝ The Hahn and Metcalfe conclusion is pretty robust.
  - ⇝ In fact, uncertainty in the non-CARE elasticity is not enough to break their result.

Extensions to the basic model

Our welfare bounds for the basic model rely on a number of modeling choices:

1. *In practice, the analyst might make assumptions about demand curvature.*

   $\Longrightarrow$ We show how **demand curvature** assumptions lead to tighter bounds.

2. *In practice (e.g., counterfactuals), the analyst might observe p_0, p_1, and q_1, but not q_0.*

   $\Longrightarrow$ We show how to **extrapolate** from fewer observations.

3. *In practice, the analyst might observe more points on the demand curve.*

   $\Longrightarrow$ We show how to **interpolate** with more observations. Details

4. *In practice, the analyst might be limited by sampling error.*

   $\Longrightarrow$ We show how to incorporate **sampling error** into welfare bounds. Details

Further extensions: welfare beyond ΔCS

#1. Producer surplus works just as well as CS.

#2. Can handle heterogeneity + distributional questions.

#3. Can handle alternative welfare measures like EV and CV.

#4. Can handle multiple objectives at once.

- ⇝ E.g., Pareto-weighted consumer surplus + DWL.

▸ Skip to the end

Summing up

- **This paper.** Develops a framework to bound welfare based on economic reasoning.
- **Building on previous work.** Hope to make the case that everyone should use this.
- **Use cases.** Draw/assess conclusions from empirical objects commonly estimated.
- **Future work.** We're excited about this.
  - Robustness for structural IO-style problems (e.g., inference with endogenous pricing, merger screens, welfare in horizontally differentiated good markets).
  - Robustness for new goods and price indices (e.g., the CPI).
  - Robustness for larger macro models (e.g., extending ACR, ACDR).

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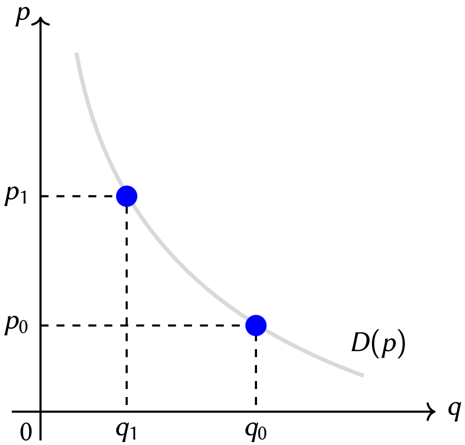
Theorem 2a. (concave-like assumptions).

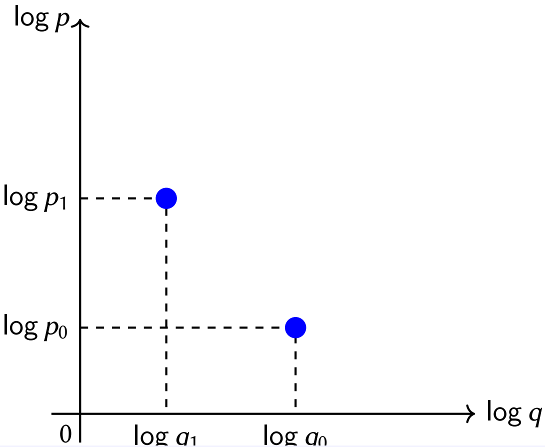
The **lower** bound for the change in consumer surplus are attained by:

(A1) decreasing elasticity: a *CES* interpolation. $D(p) = \theta_1 p^{-\theta_2}$

▸ Geometric Intuition ▸ Fancy Proof ▸ Skip Proof

Marshall's second law (decreasing elasticity) $\iff$ $\log q$ is concave in $\log p$.





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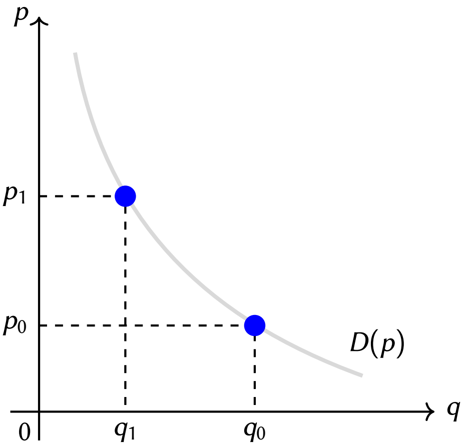

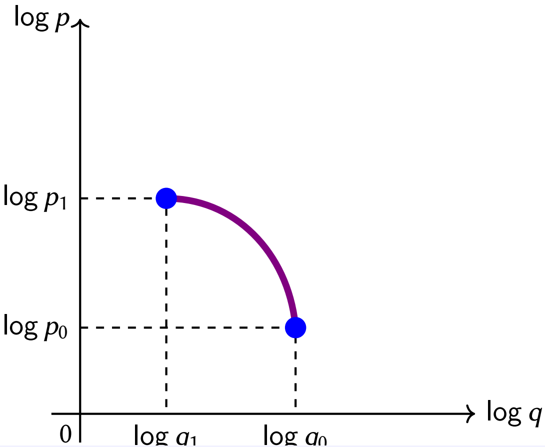

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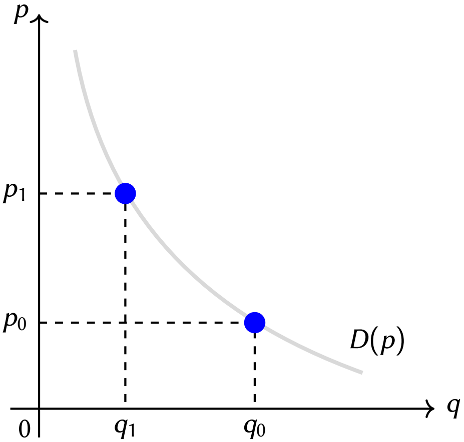

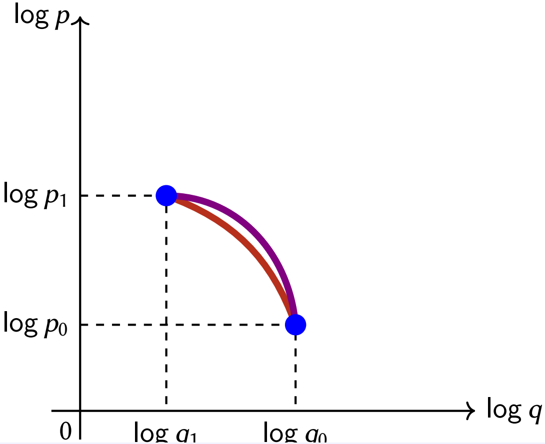

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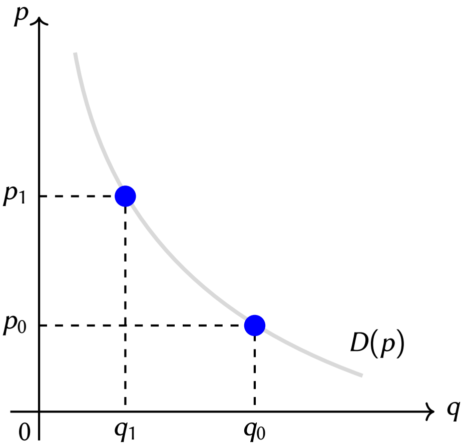

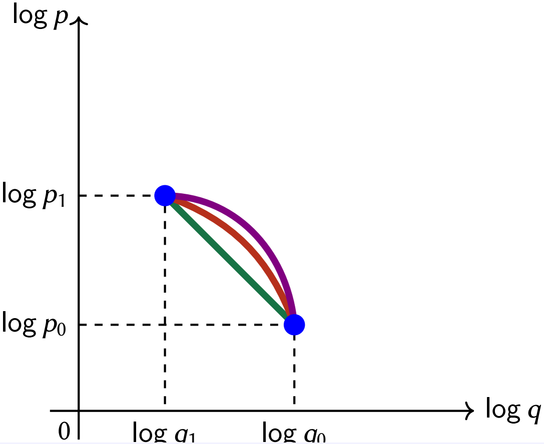

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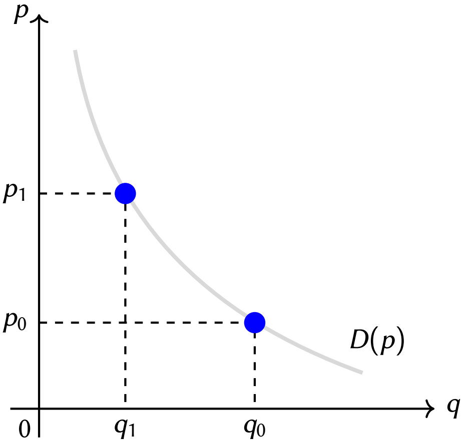

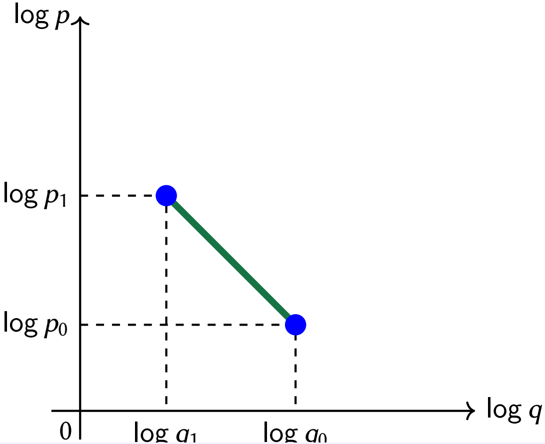

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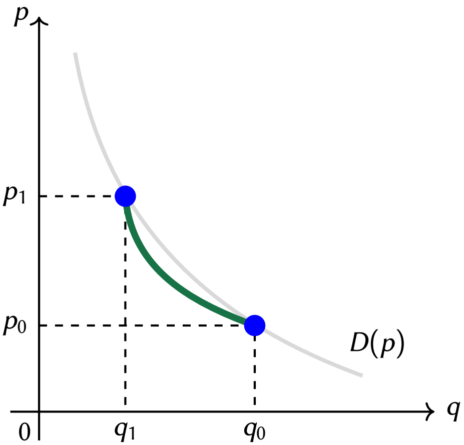
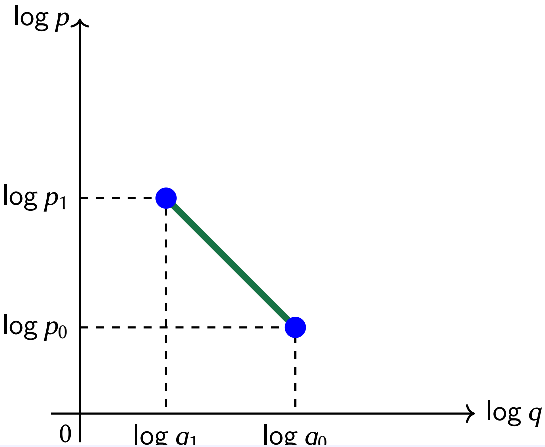




Information Design (Alternative) Proof

Back

- **Step #1.** Transform the problem.

  For each Ai, *map* $D(p)$ *to a measure* $h(p)$ *in the appropriate functional space.*

- **Step #2.** Show that welfare is "monotone" with respect to $h(p)$ under a partial order.

  Mean-preserving spreads of $h(p)$ *increase welfare.*

- **Step #3.** Derive the upper and lower bounds in terms of $h(p)$ and map back to $D(p)$.

  Lower bound is attained when $h(p)$ *is a step function (i.e., has 2 constant pieces).*

  Upper bound is attained when $h(p)$ *is constant (i.e., has 1 constant piece).*

Alternative Proof: Step #1 – Change of Variables

(A1) Decreasing Elasticity

Variable change:

$$h(\pi) := \varepsilon(e^{\pi}), \quad \text{where } \pi = \log p.$$

Mapping:
$D(p) = q_0 \exp\left[\int_{\log p_0}^{\log p} h(\pi)\,\mathrm{d}\pi\right].$

Transformation:

$$\begin{cases} \overline{\Delta CS} = q_0 \cdot \max_{h \in \mathcal{E}} \displaystyle\int_{p_0}^{p_1} \exp\left[\int_{\log p_0}^{\log p} h(\pi)\,\mathrm{d}\pi\right] \mathrm{d}p, \\ \underline{\Delta CS} = q_0 \cdot \min_{h \in \mathcal{E}} \displaystyle\int_{p_0}^{p_1} \exp\left[\int_{\log p_0}^{\log p} h(\pi)\,\mathrm{d}\pi\right] \mathrm{d}p. \end{cases}$$

(A6) Convex Demand

Variable change:

$$h(p) := D'(p).$$

Mapping: $D(p) = D(p_0) + \int_{p_0}^{p} h(s)\,\mathrm{d}s.$

Transformation:

$$\begin{cases} \overline{\Delta CS} = \max_{h \in \mathcal{E}} \displaystyle\int_{p_0}^{p_1} (p_1 - p)\,h(p)\,\mathrm{d}p, \\ \underline{\Delta CS} = \min_{h \in \mathcal{E}} \displaystyle\int_{p_0}^{p_1} (p_1 - p)\,h(p)\,\mathrm{d}p. \end{cases}$$

Example: (A6) Convex Demand

Definition: $h_2 \succeq h_1$ if h_2 is a mean-preserving spread of h_1

$$h_2 \succeq h_1 \iff \int_{p_0}^{p} h_2(s)\, \mathrm{d}s \geq \int_{p_0}^{p} h_1(s)\, \mathrm{d}s \qquad \forall\, p \in [p_0, p_1].$$

- This defines a *partial order* on the family of $h(p)$
  - $\Rightarrow$ Can think of this as second-order stochastic dominance
  - $\Rightarrow$ For (A6), think of $h(p)$ as a CDF: increasing with a mean constraint:

$$D(p_0) = q_0 \quad \text{and} \quad D(p_1) = q_1 \implies \int_{p_0}^{p_1} h(p)\, \mathrm{d}p = q_0 - q_1.$$

Example: (A6) Convex Demand

Lemma. The welfare objective is monotone in the partial order $\succeq$:

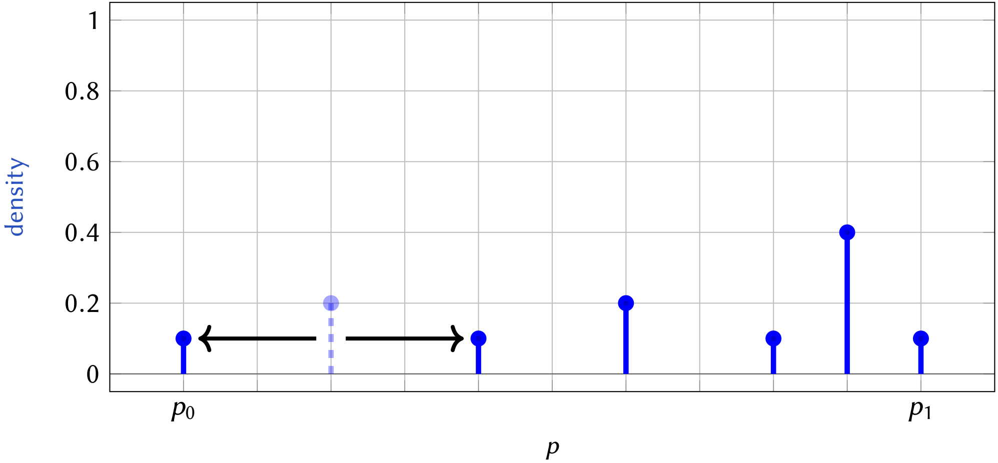
$$h_2 \succeq h_1 \implies \int_{p_0}^{p_1} (p_1 - p)\, h_2(p)\, \mathrm{d}p \geq \int_{p_0}^{p_1} (p_1 - p)\, h_1(p)\, \mathrm{d}p.$$

Intuition: Risk-averse gamblers prefer contractions of lotteries

Corollary. The upper (*resp.*, lower) bound is attained by iteratively applying mean-preserving spreads (*resp.*, mean-preserving contractions) to $h(p)$.

Step #3: deriving the *upper* bound

Consider the density that generates $h(p)$, where $h(p)$ is viewed as a CDF:



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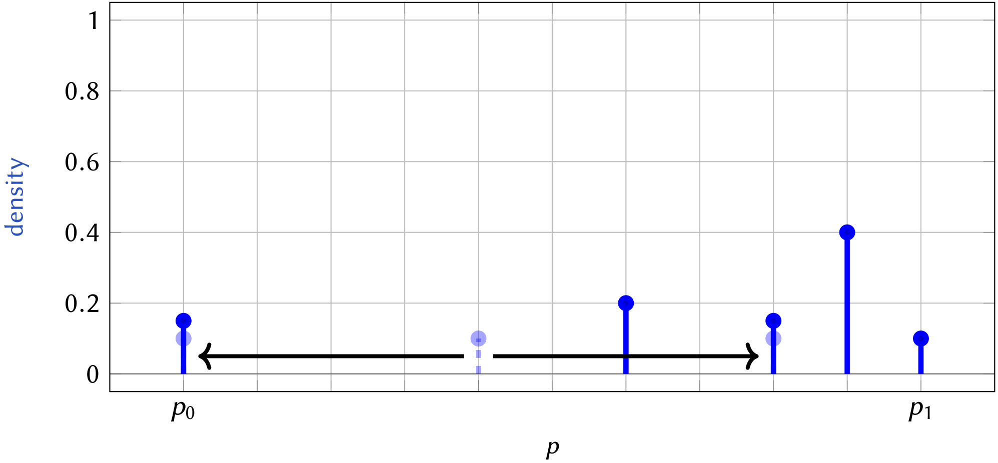

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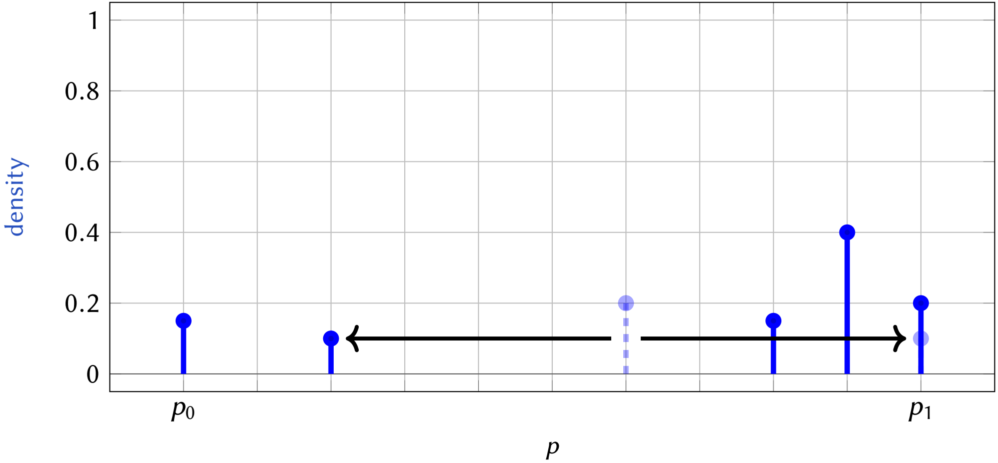

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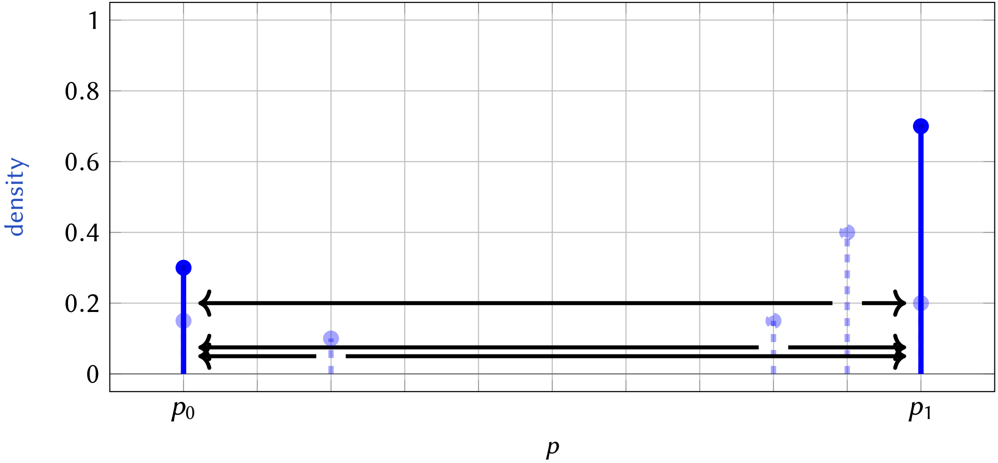

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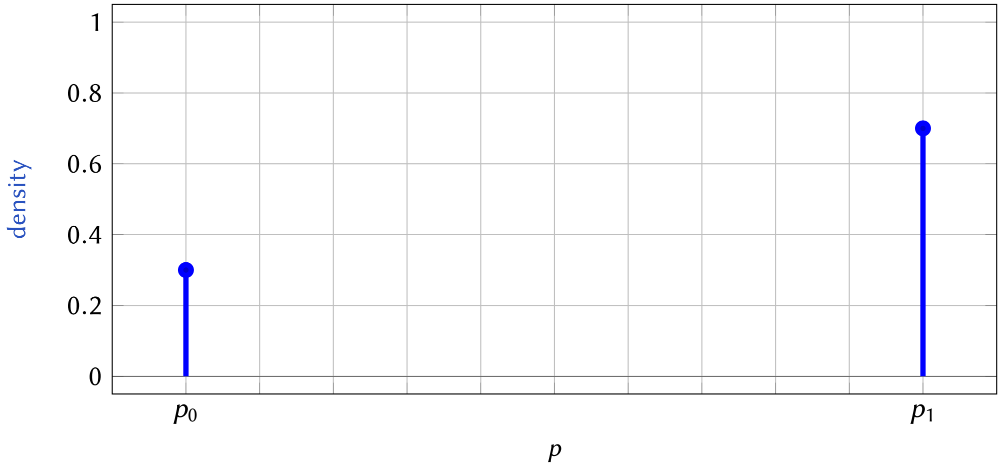
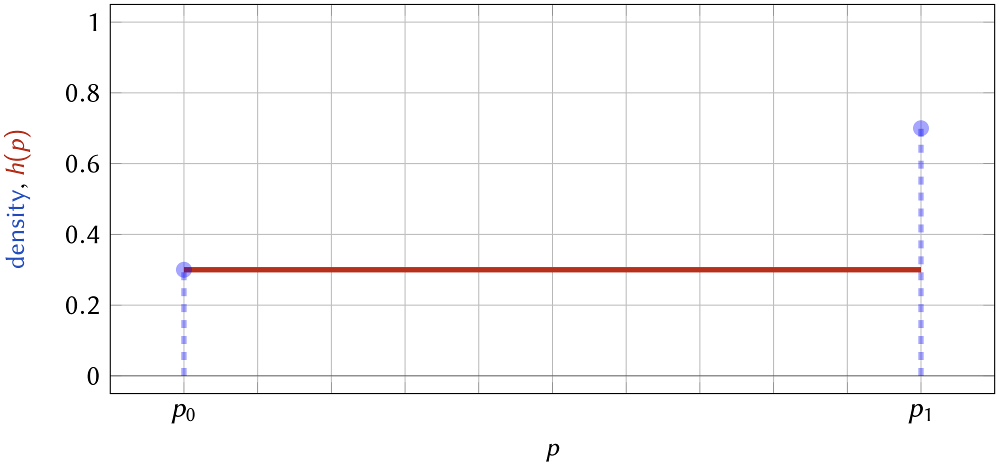


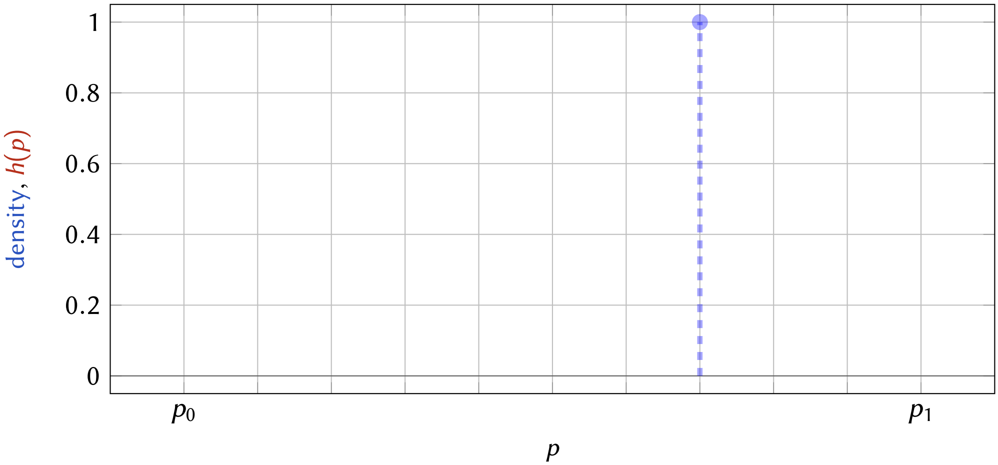
Step #3: deriving the *upper* bound

So the $h(p)$ that attains the **upper bound on welfare** is **constant** between p_0 and p_1:



Step #3: deriving the *lower bound*

Similarly, the $h(p)$ that attains the **lower bound on welfare** is a **step function**.



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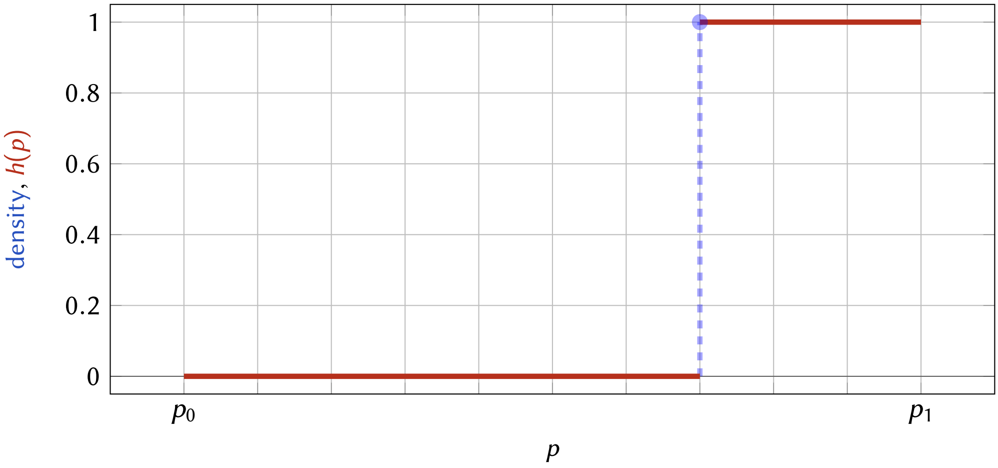


Step #3: deriving welfare bounds

Back

- Mapping back from $h(p)$ into demand curves $D(p)$:

$$h(p) \text{ is constant} \iff D'(p) \text{ is constant} \iff D(p) \text{ is linear.}$$

Step #3: deriving welfare bounds

- Mapping back from $h(p)$ into demand curves $D(p)$:

$$h(p) \text{ is constant } \iff D'(p) \text{ is constant } \iff D(p) \text{ is linear.}$$

- This proves the bounds for assumption (A6) (convexity of demand):
  - The **upper bound** is attained by a 1-piece linear interpolation.
  - The **lower bound** is attained by a 2-piece linear interpolation.

Step #3: deriving welfare bounds

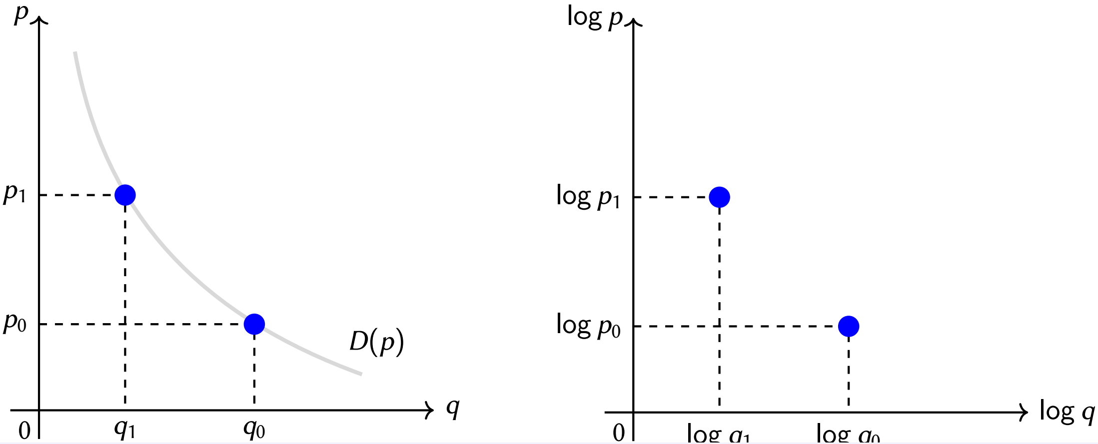
Back

- Mapping back from $h(p)$ into demand curves $D(p)$:

$$h(p) \ \text{is constant} \iff D'(p) \ \text{is constant} \iff D(p) \ \text{is linear}.$$

- This proves the bounds for assumption (A6) (convexity of demand):
  - The **upper bound** is attained by a 1-piece linear interpolation.
  - The **lower bound** is attained by a 2-piece linear interpolation.
- The same proof strategy works for all the other assumptions (with different $h(p)$).

Marshall's second law (decreasing elasticity) + elasticity lies in $[\underline{\varepsilon}, \overline{\varepsilon}]$



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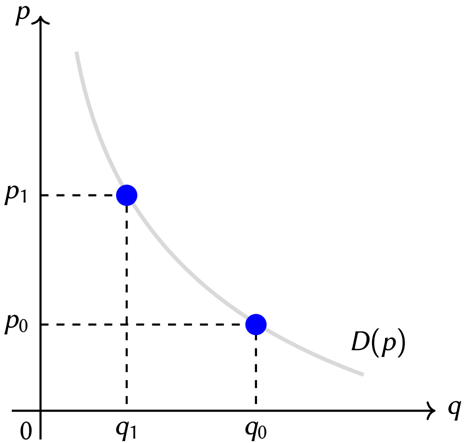

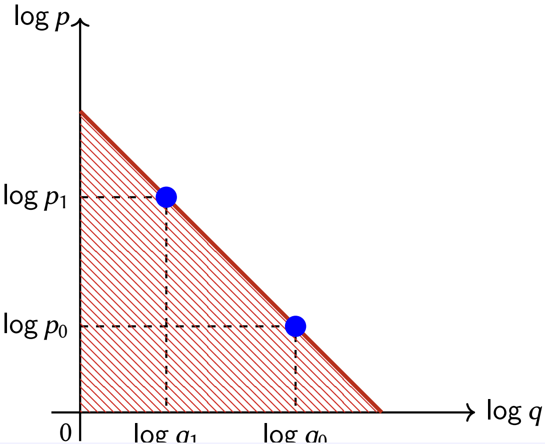

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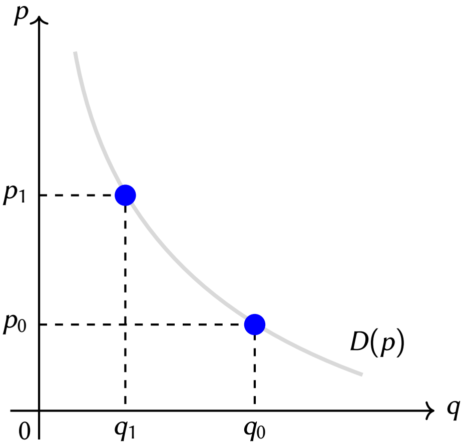

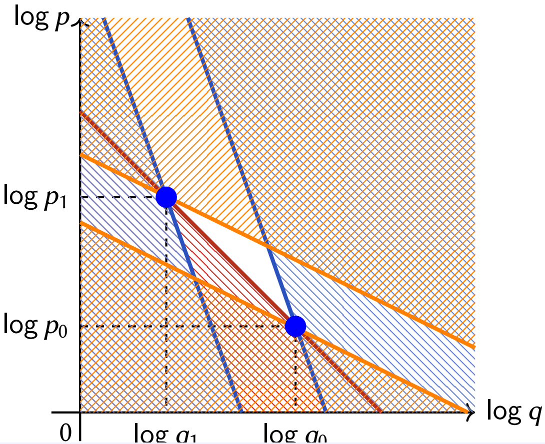

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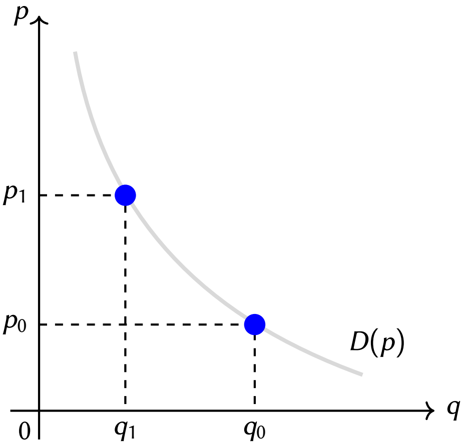

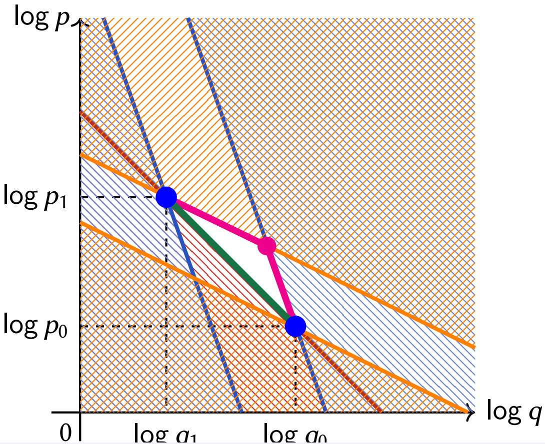

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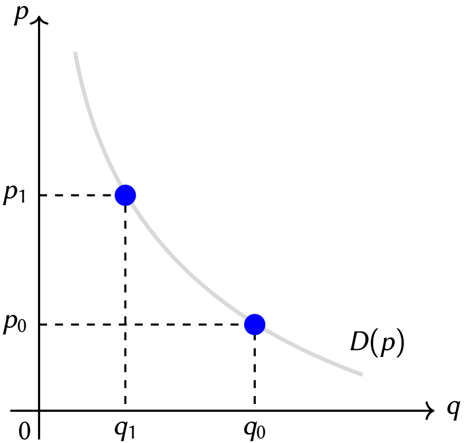

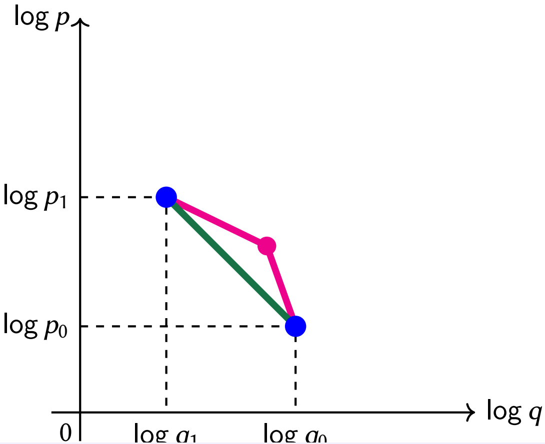

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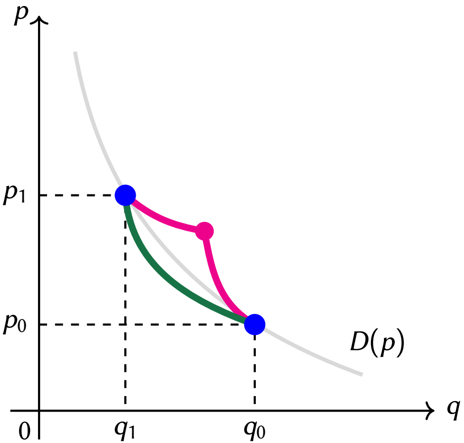
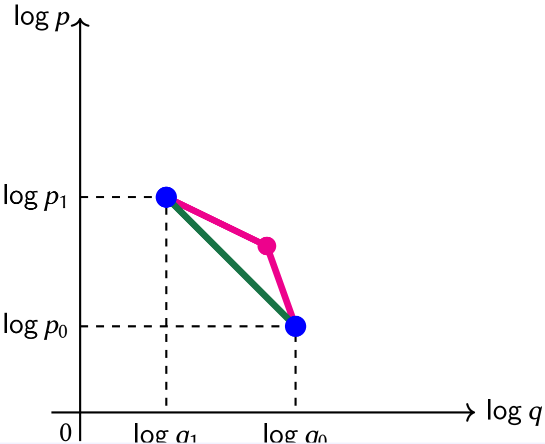
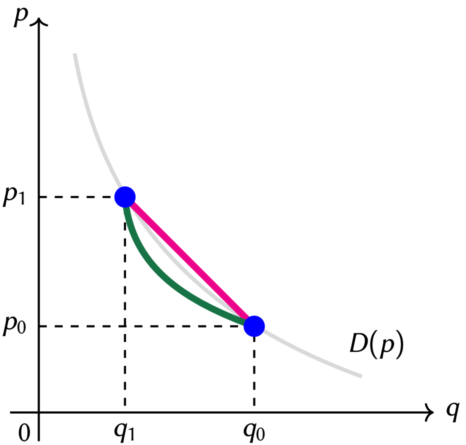




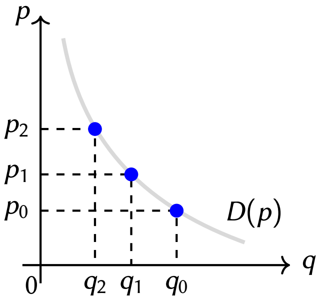
Marshall's second law (decreasing elasticity) + convex demand

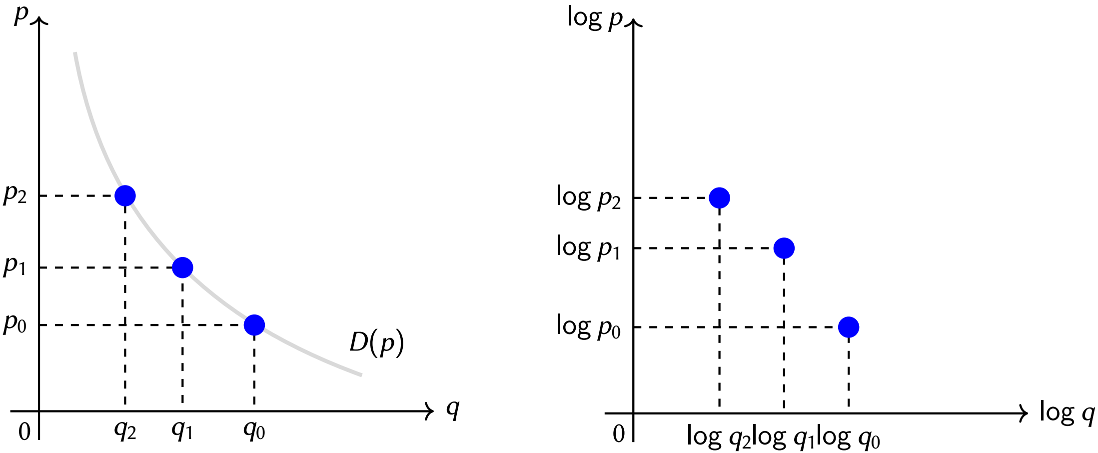


An analyst observes **3 points** on a demand curve: (p_0, q_0), (p_1, q_1), and (p_2, q_2).

We assume that elasticity between p_0 and p_2 lie in the interval $[\underline{\varepsilon}, \overline{\varepsilon}] \subset \mathbb{R}_{\leq 0}$.

Question. What is the change in consumer surplus from p_0 to p_2?

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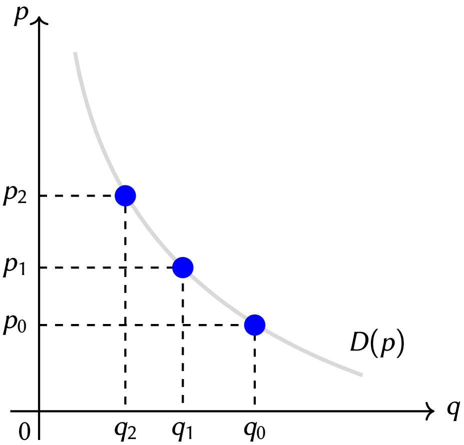

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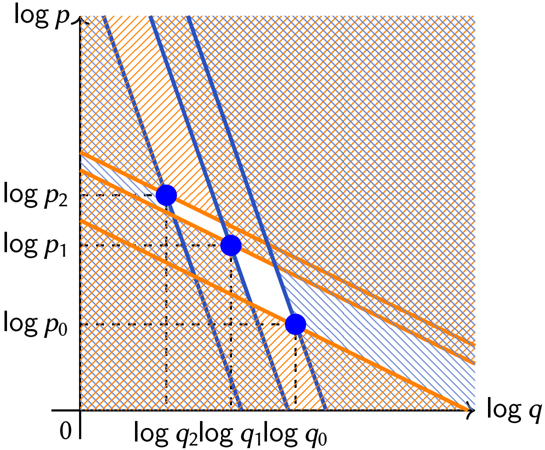

$\log p$
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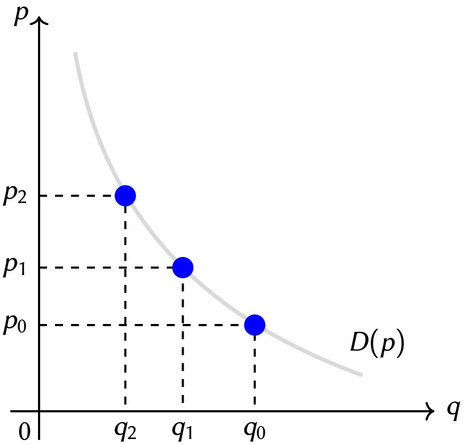
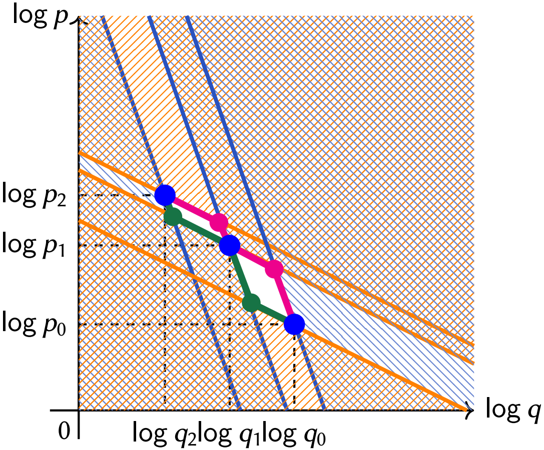
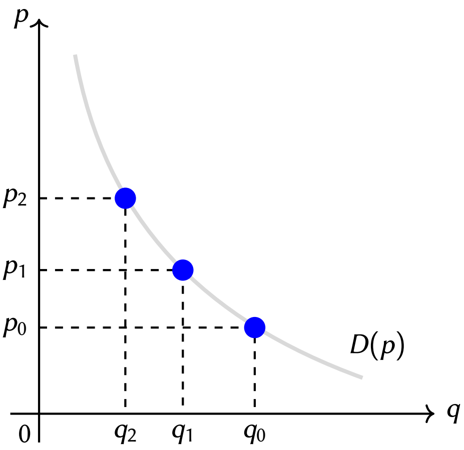
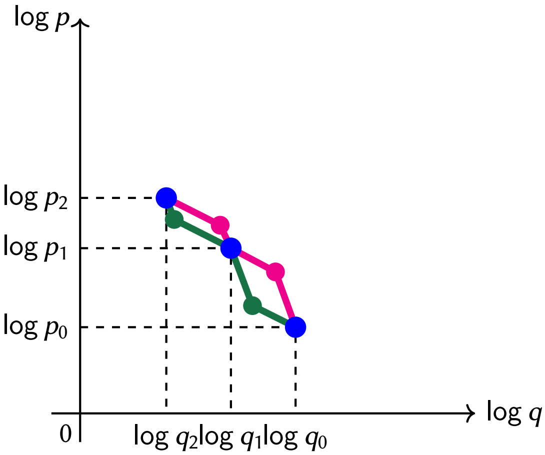
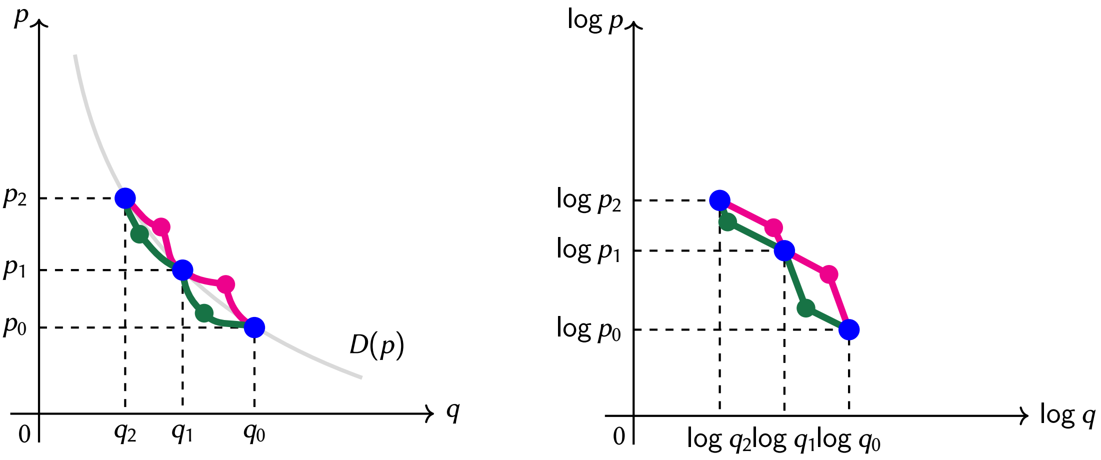
0
$\log q_2$ $\log q_1$ $\log q_0$
$\log q$


p
p2
p1
p0
D(p)
q
0
q2
q1
q0


log p
log p2
log p1
log p0
log q
0
log q2
log q1
log q0


p
p_2
p_1
p_0
$D(p)$
q
0
q_2
q_1
q_0


log p
log p_2
log p_1
log p_0
log q
0
log q_2
log q_1
log q_0


p
p2
p1
p0
0
q2
q1
q0
q
D(p)
log p
log p2
log p1
log p0
0
log q2log q1log q0
log q

Quantities demanded might be noisily observed:

$$q_1 = D(p_1) + e \quad \text{where} \quad e \sim \mathcal{N}\left(0, \sigma^2 / N_1\right) \tag{1}$$

Question. What is the 95% CI on the change in consumer surplus from p_0 to p_1?

⇒ The bounds $\overline{\Delta CS}(q_0, q_1)$ and $\underline{\Delta CS}(q_0, q_1)$ are monotonic in q_1

⇒ Obtain CIs by plugging in the CIs of q_1