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A Note on the Equilibrium Auction for Contract Bidding

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I. Introduction

A recent paper by Charles Holt[1/] compared expected price outcomes in contract bidding using discriminating vs. competitive auctions.[2/] The major result was to show that, if bidders are risk averse (risk neutral), the expected contract price in the discriminating auction is less than (equal to) the expected contract price in the competitive auction.[3/] Since some degree of risk aversion on the part of bidders is plausible, these results suggest that the discriminating auction would be preferred by the buyer; however, this conclusion remains tentative for two reasons.

The first is that the choice of the bidding mechanism may have a "participation effect." That is, the choice of the bidding

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1/ "Competitive Bidding for Contracts under Alternative Auction Procedures," 88 Journal of Political Economy (June 1980) 433.

2/ "Discriminating" auction refers to the familiar seal bid auction, while "competitive" auction refers to either the open oral (English) auction or the second price sealed bid auction.

3/ In this note, I dismiss as uninteresting the possibility that bidders (or the buyer) may be risk lovers. If bidders are risk lovers then Holt's model yields the result that the expected price in the discriminating auction is greater than the expected price in the competitive auction.

Analogous results have been given for the case of the high bid auction where bidders compete to purchase rather than sell. See J. G. Riley and W. F. Samuelson, "Optimal Auctions," 71 American Economic Review (June 1981) 381 (See p. 388-89); M. Harris and A. Raviv, "Allocation Mechanisms and the Design of Auctions," 50 Econometrica (September 1982) 1089; and, for a review of established results of "received theory", see Section 2 of P. R. Milgrom and R. J. Webber, "A Theory of Auctions and Competitive Bidding," 50 Econometrica (September 1982) 1089.

system may have an indirect effect on price by attracting different numbers of competing bidders. This question is not addressed in Holt's paper, nor in the more extensive literature dealing with formal models of buyer (high bid) auctions, since the number of bidders is taken as exogeneous. Holt at least recognizes the issue, however, when he notes that "[i]t would be interesting to see if these [comparative price] results depend on the assumption that the number of bidders is independent of the auction procedure selected" (p. 442).

The second reason is that the effect of risk aversion on the part of buyers has to be considered. The result that bidder risk aversion affects expected price outcomes raises the possibility that buyer risk preference may also have important implications for the analysis. If the buyer is risk averse, a lower expected price in the discriminating auction (compared to the competitive auction with the same number of bidders) is neither necessary nor sufficient to infer a buyer's preference.

These two issues -- the possibility of a differential participation effect and the role of buyer risk preferences -- are obviously related since preferences of both buyers and sellers will determine the equilibrium auction institution. In this paper I show that Holt's model yields a strong presumption that the discriminating auction will be the equilibrium institution for contract bidding.

The paper is organized as follows. The following section sketches the model of bidding behavior in the discrimination auction. Then I show that the use of the Nash bidding strategy in that auction implies that bidders are indifferent between discriminating and competitive auctions which in turn implies that there is no differential participation effect. Following that I show that a risk averse buyer will unambiguously prefer the discriminating auction so long as bidders are not risk lovers.

II. Bidding in the Discriminating Auction

Bidding behavior in the discriminating auction involves the choice of a optimal strategy when faced with uncertainty about the bids by competitors. Following the game theoretic approach of the more recent literature in this area, Holt models bid strategies as the equilibrium of a Nash game. The model assumes that bidders have identical risk aversion and identical direct costs, C. 4/ The total cost of fulfilling the contract is C plus a firm specific opportunity cost, r. The uncertainty inherent in bidding stems from uncertainty about the specific opportunity costs of competitors. However, all firms know that competitors' r values are realizations of a random variable with pdf g(r), $r \in [\underline{r}, \bar{r}]$ with associated cdf G(r) and also know the number of (exogeneously determined) participating bidders. Given the symmetry of preferences and information about the stochastic process generating r, the Nash bid strategy function, B(r), is a monotone increasing function of r with inverse I(B) = r. The

4/ Equivalently we can view all firms as facing the same ex ante uncertainty about direct cost at bid time; C could therefore be interpreted as the certainty equivalent of this uncertain cost.

monotonicity of the bid function implies that the firm with the lowest value of r has the lowest bid. Therefore, the probability that one of the N competing firms, with opportunity cost r, has a bid lower than the minimum bid of its n = N-1 competitors is $1-F_{n,1}(I(B)) = 1-F_{n,1}(r)$. $F_{n,1}(r) = 1 - [1-G(r)]^{N-1}$ is the cdf of the first order statistic (minimum value) of the random variable r in a sample of n with associated pdf $f_{n,1}(r) = (N-1)[1-G(r)]^{N-2}g(r)$.

Utilizing the Nash bid strategy, the bidder's expected utility is

$$EU_d = [1-F_{n,1}(I(B)]U(B - C) + F_{n,1}(I(B))U(r)$$

$$= [1-F_{n,1}(r)]U(B - C) + F_{n,1}(r)U(r). \tag{1}$$

The equation [(8) p. 438] which implicitly defines B(r) in this model is

$$U(B(r) - C) = \frac{\int_r^{\bar{r}} U(s)f_{n,1}(s)ds}{[1- F_{n,1}(r)]} . \tag{2}$$

B(r) is determine by applying the inverse of the utility function to (2) and adding C to both sides.

III. Bidder Indifference

Using (2) to substitute for the first term on the RHS of (1) we can now write the expected utility of a bidder in the discriminating auction as

$$(3) \quad EU_d = \int_{\underline{r}}^{\bar{r}} U(s) f_{n,1}(s) ds + F_{n,1}(\underline{r}) U(\underline{r}) .$$

We now ask whether bidders would have different expected utilities in the competitive auction. If expected utilities differ and there is a sunk cost of bidding (a bid preparation cost for example) participation would differ so the expected price comparison would have to take this effect into account. The result we now show is that expected utilities are identical.

There are two types of competitive auctions to consider, the second-price sealed bid auction proposed by Vickery[5] and the open oral, or English, auction. In the second-price auction, the contract price is determined by the second lowest bid. In this auction, each bidder's utility maximizing strategy is to bid his indifference price C+r, hence the low bidder receives a contract price equal to the indifference price of the second lowest bidder, $b = C + r_1$, where r_1 is the minimum value of r (first order statistic) in the sample of his n competitors. In the open oral (English) auction, bids are successively lowered until only one bidder remains; the low bidder in this auction also receives a contract price equal to $b = C + r_1$. Since C is a constant, b is distributed as $f_{n,1}(b - C)$ on $[\underline{r} + C, \bar{r} + C]$; so the expected utility of a bidder in a competitive auction is given by

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5/ W. Vickery, "Counterspeculation, Auctions, and Competitive Sealed Tenders," 16 Journal of Finance (March 1961) 8.

$$EU_c(r) = \int_{r+C}^{\bar{r}+C} U(s - C)f_{n,1}(s-C)ds + \int_{\underline{r}+C}^{r+C} U(r)f_{n,1}(s - C)ds$$

$$= \int_{r}^{\bar{r}} U(s)f_{n,1}(s)ds + F_{n,1}(r)U(r). \tag{4}$$

which is identical to (3). Therefore, each bidder's expected utility in the competitive auction is identical to his expected utility in the discriminating auction. Hence, we can conclude that if N bidders were invited to tender bids in one of the two auctions, no bidder would withdraw if it were then announced that the other auction would be used. In conclusion, there is no differential participation effect. This result is independent of the form of the bidders' (identical) utility functions and the form of the pdf of r. The implication of the bidder indifference result is that the equilibrium bidding institution depends only on the buyer's preference which we examine in the next section.

Before turning to that issue, it is of some interest to consider the reason for the difference in expected price outcomes in the two auctions with risk averse bidders. Holt's intuitive explanation is that "there is less uncertainty in the competitive auction. In this auction, the firm with the lowest opportunity cost learns the bids of its rivals as the bid price is being lowered. On the [other] hand, in a discriminatory auction, no firm knows its rivals' bids prior to making his own bid. The greater uncertainty in a discriminatory auction causes risk averse bidders to submit low bids..." (p. 441).

This explanation is suspect for two reasons. First, it reverses the usual implication of risk aversion since Holt is saying that bidders' response in playing the more risky game is to accept a lower expected payoff. Second, the notion that the competitive auction is more certain because bidders learn about rivals costs as the bids are lowered ignores the fact that the English auction yields the same outcome as the second-price sealed bid auction in which no information is revealed. In contrast to Holt's intuitive argument, it will be be shown that the discriminating auction is <u>less</u> risky than the competitive auction. This demonstration also provides a straightforward proof of the result that bidder risk aversion leads to a lower expected price in the discriminating auction.

From the bidder indifference result, we know that the first terms in equations (1) and (4) are equal. Dividing through by $[1-F_{n,1}(r)]$ gives equation (2) which now can be given the following interesting interpretation: conditional on winning, the utility of profit in the discriminating auction is equal to the conditional expected utility of profit in the competitive auction. In other words, equation (2) shows that B(r)-C is the certainty equivalent of the (conditional) expected utility in the competitive auction given by the RHS of (2). If bidders' utility functions exhibit risk aversion, it follows that the certainty equivalent will be less than the expected value of profits in the competitive auction; hence

$$(5) \quad B(r) - C < \frac{\int_{r}^{\bar{r}} s\, f_{n,1}(s)ds}{1 - F_{n,1}(r)}$$

Since the inequality in (5) holds for all values of r (except at the upper limit where both sides are equal to r [6/]) it will hold if we take the expectation of both sides of (5) with respect to the same density function on $[\underline{r}, \bar{r}]$. Let this density function be that of the first order statistic of r in a sample of N bidders. Then, taking expectations we have E_d(bidder profits) < E_c(bidder profits). Since C is identical for all bidders this directly implies that, with risk averse bidders, $E[B(r_1)] < E(r2) + C$, where $r1, \bar{r}2$ here denote the first and second order statistics of r in the total sample of N bidders.

The correct explanation for the lower price in the discriminating auction is thus that risk averse bidders pay for the greater (conditional) certainty of profits in the discriminating auction by bidding a profit which is lower than their expected profit in the competitive auction.

III. Risk Averse Buyers Prefer the Discriminating Auction

The lower expected contract price in the discriminating auction together with the bidder indifference between the two auctions unambiguously implies that a risk neutral buyer will prefer the discriminating auction if bidders are risk neutral. Can we say anything in general about buyer preference if the

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6/ The fact that $\lim_{r \to \bar{r}} B(r) = r + C$ can be verified by using equation (2).

buyer is risk averse? In this section I show that a buyer with a utility function exhibiting risk aversion will prefer the discriminating auction so long as bidders are not risk lovers.7/ The proof is given in two steps. In the first we show that when bidders are risk neutral, a risk averse buyer will prefer the discriminating auction. Once this is established, it is straightforward to use the results of the previous section to show that the buyer's expected utility in the discriminating auction with risk averse bidders is greater than if bidders are risk neutral.

In what follows we will be considering the buyer's expected utility when taking bids from N bidders. The pdf's and cdf's of the first and second order statistics of g(r) in a sample of N are distinguished by the subscript N: $f_{N,i}(.)$ and $F_{N,i}(.)$, i = 1,2. For clarity it will be useful to also explicitly distinguish the random variates as r_i, i =1,2, when referring to the total sample of N bidders The utility function of the buyer is denoted as V(.), and W denotes the gross value of the contract

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7/ This result is a bit surprising in that nothing can be inferred about comparative riskiness of the two price distributions from the properties of order statistics, even if we restrict the class of utility functions to those which allow riskiness to be identified with variance. The expected value of price in the discriminating auction is C plus the expected value of a function of the first order statistic, while the expected price in the competitive auction is C plus the expected value of the second order statistic. Suppose we start by provisionally identifying riskiness with variance. If r values are drawn from a rectangular distribution (commonly used in the literature for numerical examples) then $var(r_1) < var(r_2)$ which would accord with the result to be shown. But if the r values are drawn from a normal distribution, for example, then $var(r_1) > var(r_2)$.

to the buyer.

We first prove that if the buyer is risk averse and bidders are risk neutral, then $EV_d > EV_c$, that is, we will show that

$$(6) \quad \int_{\underline{r}}^{\bar{r}} V(W - B(r_1))f_{N,1}(r_1)ds \; > \; \int_{\underline{r}}^{\bar{r}} V(W - C - r_2)f_{N,2}(r_2)ds \; .$$

The method of proof is to show that the expression on the left, for EV_d, is equal to

$$(7) \quad \int_{\underline{r}}^{\bar{r}} V(W - C - E[r_2|r_1])f_{N,1}(r_1)\, dr_1 \; ,$$

and the expression on the right, for EV_c, is equal to

$$(8) \quad \int_{\underline{r}}^{\bar{r}} E[V(W - C - r_2|r_1)]f_{N,1}(r_1)dr_1 \; .$$

We can then infer (6) by Jensen's inequality.

Consider first the LHS of (6), which gives the buyer's expected utility of the contract in the discriminating auction. If bidders are risk neutral then, using a linear utility function in (2), the winning bid in the discriminating auction is

$$(9) \quad B(r_1) = \frac{\int_{r_1}^{\bar{r}} sf_{n,1}(s)ds}{1-F_{n,1}(r_1)} + C = \frac{\int_{r_1}^{\bar{r}} s(N-1)[1- G(s)]^{N-2}g(s)ds}{[1-G(r_1)]^{N-1}} + C.$$

The conditional density of r_2 given r_1 is[8/]

$$\frac{f_{N,1,2}(r_1,r_2)}{f_{N,1}(r_1)} = \frac{N!/(N-2)![1-G(r_2)]^{N-2}g(r_2)g(r_1)}{N!/(N-1)!\ [1-G(r_1)]^{N-1}g(r_1)},$$

therefore,

$$(10)\quad E(r_2|r_1) = \frac{\int_{r_1}^{\bar{r}} r_2(N-1)[1-G(r_2)]^{N-2}g(r_2)dr_2}{[1-G(r_1)]^{N-1}},$$

which is identical to the first term on the RHS of (9), so the buyer's expected utility in the discriminating auction can be expressed as (7).

Now consider the RHS of (6). Writing out the full expression for $f_{N,2}(\ )$, buyer expected utility in the competitive auction is given by

$$(11)\quad EV_c = \int_{\underline{r}}^{\bar{r}} V(W - C - r_2)N(N-1)[1-G(r_2)]^{N-2}G(r_2)g(r_2)dr_2$$

$$= N(N-1)\int_{\underline{r}}^{\bar{r}} V(W - C - r_2)[1-G(r_2)]^{N-2}\int_{\underline{r}}^{r_2} g(x)dx\ g(r_2)dr_2\ ;$$

changing the order of integration yields

$$= N(N-1)\int_{\underline{r}}^{\bar{r}} g(x) \int_{x}^{\bar{r}} V(W - C - r_2)[1-G(r_2)]^{N-2}\ g(r_2)dr_2dx.$$

8/ See R. V. Hogg and A. T. Craig, Introduction to Mathematical Statistics, 3rd ed. (New York: Macmillian, 1970) 145 ff.

Now multiply and divide by $[1-G(x]^{N-1}$ to get

$$= \int_{\underline{r}}^{\bar{r}} N[1-G(x)]^{N-1}g(x) \frac{(N-1)\int_{x}^{\bar{r}} V(W-C-r_2)[1-G(r_2)]^{N-2}g(r_2)dr_2}{[1-G(x)]^{N-1}} dx$$

$$(12) \qquad = \int_{\underline{r}}^{\bar{r}} E[U(W - C - r_2 | r_1)f_N(r_1)dr_1$$

as given by (8). If V( ) exhibits risk aversion then, by Jensen's inequality, $EV_d = E[V(W-C-E(r_2|r_1))] > E[E(W-C-r_2|r_1)] = EV_c$. This proves that a risk averse buyer prefers the discriminating auction when bidders are risk neutral.

The second part of the proof is to show that expected utility of the buyer is also greater in the discriminating auction when bidders are risk averse. This follows from the fact that bid for any value of r, and thus that of the bidder with $r = r_1$, is lower if bidders are risk averse rather than risk neutral.

V. Conclusion.

In this note I have shown that Holt's contract bidding model yields the following two implications. 1) Bidders are indifferent between the discriminating (first-price) auction and the competitive (second-price or English) auction. If N bidders accept invitations to bid in one, none would withdraw if prior to the bidding the buyer announced that the other auction were to be used instead. Thus, while the model takes the N as fixed, the qualitative comparison of the outcomes is not affected by this assumption. Bidder indifference between the two types of auctions further implies that the equilibrium bidding institution can be determined by buyer preferences taking N as fixed since there is no differential participation effect. 2) Buyer preference for the discriminating auction, with N fixed, is an implication of buyer risk aversion if bidders are either risk neutral or risk averse. Together these two results yield a fairly strong presumption that the discriminating auction will be the equilibrium contract bidding institution. Ignoring the possibility that either the buyer or bidders are risk lovers, the implication is ambiguous if and only if both parties are risk neutral.