chosen firm is competent. We again only need to consider one-shot deviations, i.e., for all histories $h \in \mathcal{H}_b^{\text{col}}$ and all types of the other firm ($\theta \in \{C, I\}$), the continuation payoff following investment should be greater or equal than that following a single deviation. Let us denote the present discounted expected equilibrium profit of a competent firm under branding with a θ-type firm after history $(h, d) \in \mathcal{H}_i^{\text{col}}$ by $V(h, d; \theta)$ and the continuation payoff after no investment (assuming the firm follows the equilibrium strategy after the deviation) by $\tilde{V}(h, d; \theta)$. Then, a **RE** exists if and only if $V(h, d; \theta) \geq \tilde{V}(h, d; \theta)$ for all h, d, θ. The equivalent to Lemma 1 is the following.

Lemma 3. *In the case of collective reputation, the* **RE** *exists if and only if*

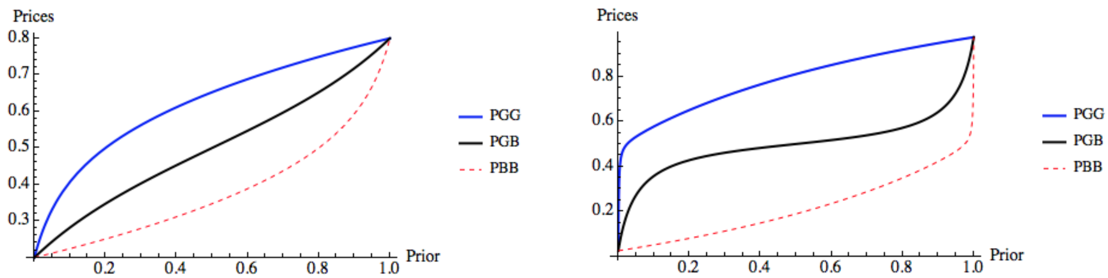
$$c \;\; \leq \;\; \hat{c}(\mu, \pi_H, \pi_L) \equiv \delta \cdot \frac{\Delta\pi}{2} \cdot \min_{h_1 \in \{G,B\}, \theta \in C,I} \hat{d}(h_1, \theta) \tag{6}$$

where

$$\begin{aligned} \hat{d}(h_1, \theta) \;\; = \;\; & \underbrace{p(h_1 G) - p(h_1 B)}_{\textit{short-run benefits}} \\ & + \delta \Big(\underbrace{\frac{\pi_H + \pi(\theta)}{2} (p(GG) - p(BG)) + (1 - \frac{\pi_H + \pi(\theta)}{2})(p(GB) - p(BB))}_{\textit{long-run benefits}} \Big), \end{aligned} \tag{7}$$

and $\pi(\theta) = \pi_H$ *if* $\theta = C$ *and* π_L *if* $\theta = I$.

Analogously to the case of individual firms, $\hat{d}(h_1, \theta)$ summarizes the future benefit of investment when the group's most recent outcome is h_1 and the type of the other firm is $\theta \in \{C, I\}$. If the short-run benefit $p(h_1 G) - p(h_1 B)$ is small, the firm's incentive to invest is low and it wants to *free-ride on its current reputation*. The long-run benefit to be realized in period $t + 2$ depends on the outcome generated by the group in period $t + 1$. If the group produces a good outcome, the firm would enjoy a price premium $p(GG) - p(BG)$. This event occurs with probability $\frac{\pi_H + \pi(\theta)}{2}$ because each firm is visited equally likely. The group produces a bad signal in period $t + 1$ with the remaining probability and receives a



(a) Small $\Delta\pi$: $\pi_H = 0.8,\ \pi_L = 0.2$ (b) Large $\Delta\pi$: $\pi_H = 0.975,\ \pi_L = 0.025$

Figure 4: Prices as a function of prior μ

premium of $p(GB) - p(BB)$. Therefore, the long-run benefits capture the *free-riding on future investments by the collective*. An individual brand only produces a good outcome in period $t+1$ with a probability $\frac{\pi_H}{2}$. However, a collective brand does with a greater probability $\frac{\pi_H + \pi(\theta)}{2}$ because the other firm can produce a good signal, too. In the next section we compare two regimes of reputation in detail and show how collective reputation can provide better incentives for investment despite this free-riding incentives.

Next, we identify the pair $(h_1, \theta) \in \{G, B\} \times \{C, I\}$ that minimizes $\hat{d}(\cdot, \cdot)$. First, for any given θ, $h_1 = G$ is binding if and only if

$$p(GG) - p(GB) \leq p(BG) - p(BB), \tag{8}$$

Moreover, if (8) then $\theta = C$ is binding because it places a higher probability on $h_1 = G$. Thus, $(h_1, \theta) = (G, C)$ attains the minimum for $\hat{d}$ in that case, and otherwise (B, I) does. By plugging in the beliefs for prices, we obtain the following characterization of $\hat{d}$:

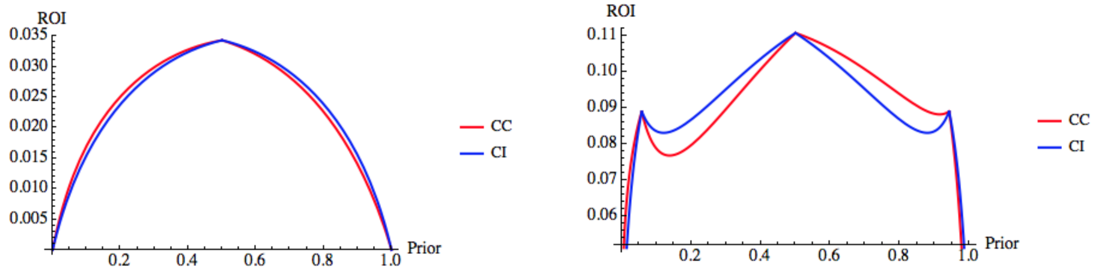
Lemma 4. *(i) If $\Delta\pi$ is small, there exists $\mu_1 \in (0,1)$ such that $\arg\min_{h_1 \in \{G,B\}, \theta \in G,B} \hat{d}(h_1; \theta) = (G, C)$ if and only if $\mu \geq \mu_1$ and $\arg\min_{h_1 \in \{G,B\}, \theta \in G,B} \hat{d}(h_1; \theta) = (B, I)$ if and only if $\mu < \mu_1$. (ii) If $\Delta\pi$ is large, there exists μ_2, μ_3, μ_4 such that $0 < \mu_2 \leq \mu_3 \leq \mu_4 < 1$ such that $\arg\min_{h_1 \in \{G,B\}, \theta \in G,B} \hat{d}(h_1; \theta) = (G, C)$ if and only if $\mu \in [\mu_2, \mu_3) \cup [\mu_4, 1]$, and (B, I) if and only if $\mu \in [0, \mu_2) \cup [\mu_3, \mu_4)$.*

Lemma 4 identifies sufficient conditions under which either of two environments–(G, C)

or (B, I)–provides the binding constraint for the cutoff, $\hat{c}$. Figure 4 contains plots of three prices, $p(GG)$, $p(GB)$, $p(BB)$, for a small and large $\Delta\pi$. If $\Delta\pi$ is small, then signals relatively uninformative, and thus, the prior belief plays a dominant role in shaping consumers' beliefs, which then determine prices. Recall that (8) implies that the group does not find obtaining the best history GG as attractive as avoiding the worst history BB. Thus, the optimistic environment (G, C) attains the cutoff level if and only if consumers' prior is sufficiently high, and otherwise (B, I) does.

Even for a large $\Delta\pi$, at extreme values of μ, the binding environment is the same as with a small $\Delta\pi$. That is, for a very large μ (close to 1), (G, C) attains the minimum return on investment, while for a very small μ (close to 0) (B, I) does. This is because μ is relatively more informative, and hence, plays a dominant role in shaping consumers' posterior beliefs.

The simple monotonic characterization in μ breaks down in the intermediate range of the prior belief as is illustrated in Figure 5 which depicts the return on investment for large and small $\Delta\pi$. We find that in an intermediate-low range, $[\mu_2, \mu_3)$, the optimistic environment (G, C) attains the minimum level of return on investment, while in an intermediate-high range, $[\mu_3, \mu_4)$, the pessimistic environment (B, I) does. In the intermediate-low region, consumers' initial beliefs are mostly placed on the group's quality being either the lowest (II) or mixed (CI), and almost none on the best quality (CC). Therefore, a competent firm's investment decision hinges on whether an additional investment will move consumers' beliefs away from II and towards CI. With accurate signals, each signal is indicative of the corresponding firm's type. Therefore, to prove to consumers that the group is of mixed quality, the group needs just one good outcome. Thus, the investment incentive is the lowest for (G, C). Analogously if μ is in an intermediate-high range, consumers' initial belief is divided between CC and CI, and the group's desire to convince consumers that they are a group of two competent firms drives its investment incentive. Then, the group really needs all histories to be good. In an environment where this is improbable, a competent firm is discouraged from investing. Therefore, (B, I) provides the minimum return on investment.



(a) Small $\Delta\pi$: $\pi_H = 0.8,\ \pi_L = 0.2$ (b) Large $\Delta\pi$: $\pi_H = 0.975,\ \pi_L = 0.025$

Figure 5: Return on Investment under (G, C) or (B, I) as a function of prior μ

Note that this analysis shows that it is not straight forward to compare collective reputation building with the individual one. Nevertheless, we can derive some economically interpretable results in the next section.

3.3 Comparing Individual and Collective Brands

We have examined two regimes of reputation and identified conditions under which the **RE** exists. In this section, we compare the two and investigate which regime provides better incentives for the following two different limiting signal structures:

1. **Exclusive knowledge** ($\pi_L \approx 0$): If $\pi_L = 0$, then without investment a firm cannot produce a good product. Thus, a good history completely reveals that the firm is competent. If competence represents the possession of a special technology or some advanced expertise, such as watches, automobiles, electronics, this seems to be a reasonable approximation.

2. **Quality control** ($\pi_H \approx 1$): If $\pi_H = 1$, then a firm always produces a good product if it invests, which implies that one bad outcome completely reveals that the firm is incompetent. If competence is about the ability to conduct quality control, such as manufacturing of generic products, e.g., clothes or skews, this seems to be a reasonable approximation.

Note that in these two extreme cases, the fact that one observation fully reveals competence

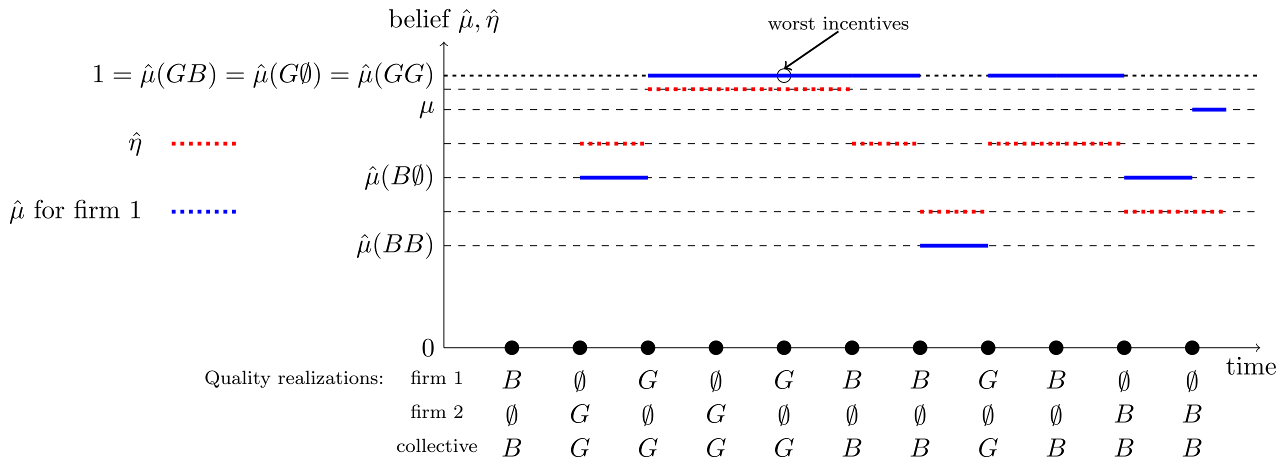


Figure 6: Belief realization for $\pi_L = 0$

or incompetence of a firm is bad for incentives of competent firms to overcome the moral hazard problem. We show under which conditions collective brands can help the firm to commit to invest.

3.3.1 Exclusive knowledge ($\pi_L \approx 0$)

If π_L is close to 0, the short-run benefit of investment in (3) after a history $h_1 = G$ vanishes for individual reputation: $p^{\text{ind}}(GG) - p^{\text{ind}}(GB) \rightarrow_{\pi_L \rightarrow 0} \pi_H - \pi_H = 0$. In contrast, under collective reputation, consumers remain uncertain about the group's quality as they belief that with positive probability only one firm is competent.

Figure 6 illustrates how beliefs evolve over time given the history realization presented at the bottom of the graph. The joint quality realizations of the two firms are outlined in the lowest row, while the realizations by individual firms are set out in the rows above. The solid blue line represents the beliefs of consumers if firm 1 builds reputation by itself. In that case, after the history $\emptyset G$ the belief must remain 1 independently of the realization in that period. This point, where the short-run benefit of investment of an individual firm vanishes, is marked by a circle.

The question is when does this benefit of collective reputation building outweigh the cost of free-riding on future investments of the other firm and of having less precise signals. Note that less precise signals lead to lower prices for a competent firm after a good signals because the customer cannot distinguish between the two firms.

Let us consider the marked history G with the worst incentives. If the prior μ is relatively optimistic, as in the graph, then the posterior $\hat{\eta}$ at this point is already very high even for a brand. However, a bad realization increases the belief that the collective brand is of type CI which causes $\hat{\eta}$ to drop significantly. Thus, in a collective the incentives of investment are much better. On the contrary, if the prior μ was very low, then even after the history G, buyers would place a high belief on the brand being of type CI, so that the cost of generating a bad quality product in the period after is relatively small.

The following proposition makes this observation formal. We show that the benefit of collective brands outweighs the free-riding cost and cost of weaker signal if the prior about the firm being competent is relatively high.

Proposition 1. *Let $\pi_L = 0$. Then, the following holds:*

i) A collective brand sustains a **RE** *for higher investment costs than an individual brand if consumers' prior belief μ about the firm's type is sufficiently optimistic and δ is not too large. Formally, $\hat{c} > \hat{c}^{ind}$ for sufficiently small π_L if $\delta \leq \frac{1}{3}$ and μ sufficiently close to 1.*

ii) An individual brand sustains a **RE** *for higher investment costs than a collective brand if the prior belief μ is sufficiently low. Formally, $\hat{c} < \hat{c}^{ind}$ for sufficiently small π_L and μ.*

One caveat of this result is that δ cannot be too large in order to make collective reputation a good commitment for investment. The reason for why δ cannot be too large is that it ensures that short-run incentives dominate long-run incentives. For the examples we have in mind, small δ can be a reasonable assumption if investment decisions are only made relatively infrequently.

Figure 7 depicts the return on investment for collective and individual brands if $\pi_H = 0.9$ and $\delta = 0.4$. One can see that collective reputation building dominates individual reputation building for a wide range of priors while for tractability we only show the result for large

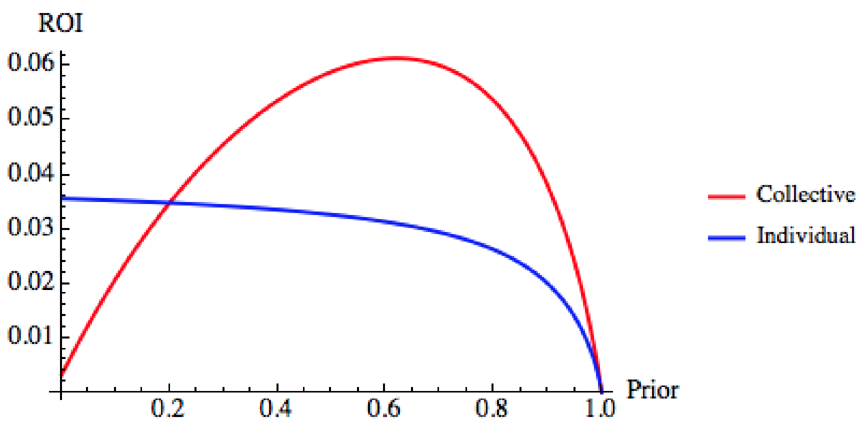


Figure 7: Comparison of Returns on Investment with $\pi_L = 0$
$\pi_H = 0.9,\ \delta = 0.4$

and small μ.

3.3.2 Quality control ($\pi_H \approx 1$)

If π_H close to 1, the short-run benefit of investment in 3 after a history $h_1 = B$ vanishes as $p^{\text{ind}}(BG) - p^{\text{ind}}(BB) \rightarrow_{\pi_L \rightarrow 0} \pi_L - \pi_L = 0$. On the contrary, for collective reputation, i.e., in 7, a bad outcome does not eliminate uncertainty entirely because the other firm may be good or bad and hence, $p(BG) - p(BB) > 0$.

Using the same history of realizations as in Figure 6, Figure 8 depicts the evolution of beliefs over time if $\pi_H = 0$ and for low prior μ. In that case, incentives for investment are the worst after a history with $h_1 = B$ which is marked by a cycle. At that point, the beliefs in the next period remain 0 independently of the quality realization today.

Again, the question is under which circumstances the benefit of collective reputation can outweigh the cost of free-riding and of having a less precise signal. Let us consider the circled history. If the prior μ is relatively pessimistic, as in the graph, then the posterior $\hat{\eta}$ is relatively low after $h_1 = B$. A good realization in the next period, however, would increase the belief that the collective brand is of type CI significantly. In contrast, if μ was small, the belief that the brand type is CI would be relatively large to start with after a history $h_1 = 1$. The following proposition is analogous to Proposition 1 and makes this observation

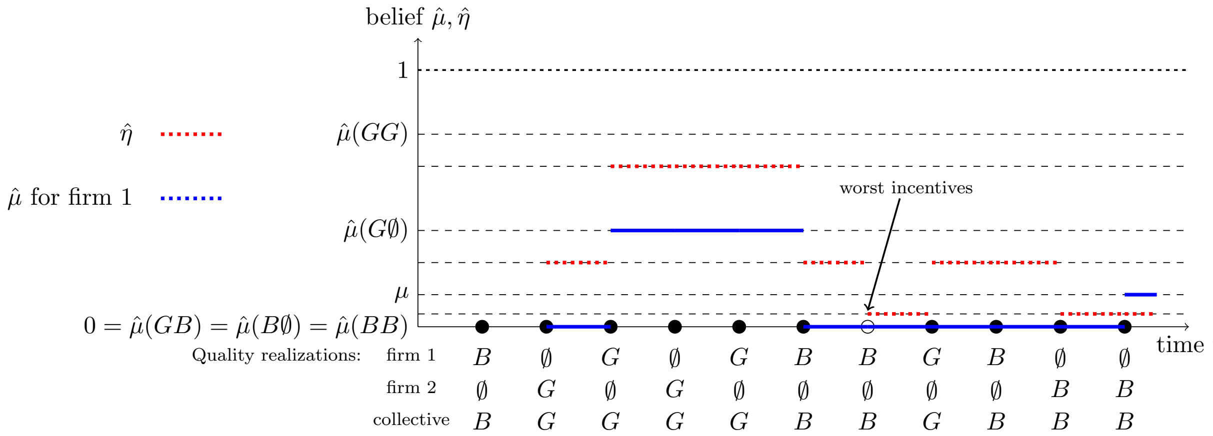


Figure 8: Belief realization for $\pi_H = 1$

formal.

Proposition 2. *Let $\pi_H = 1$. Then, the following holds:*

i) A collective brand sustains a **RE** *for higher investment costs than an individual brand if consumers' prior belief μ about the firm's type is sufficiently pessimistic and δ is not too large. Formally, $\hat{c} > \hat{c}^{ind}$ if $\delta \leq \frac{2\pi_L}{3+\pi_L}$ and μ sufficiently close to 0.*

ii) An individual brand sustains a **RE** *for higher investment costs than a collective brand if the prior belief μ is sufficiently high. Formally, $\hat{c} < \hat{c}^{ind}$ for sufficiently large μ.*

Figure 9 depicts the return on investment for collective and individual brands if $\pi_H = 1$, $\pi_L = 0.6$, and $\delta = 0.2$. One can see that collective reputation building dominates individual reputation building for a wide range of priors while for tractability we only show the result for large and small μ.

Analogously to the "exclusive knowledge" case, we need δ to be not too large for collective reputation building being a good commitment device, as otherwise the short-run benefit is dominated by the long-run free-riding incentives.

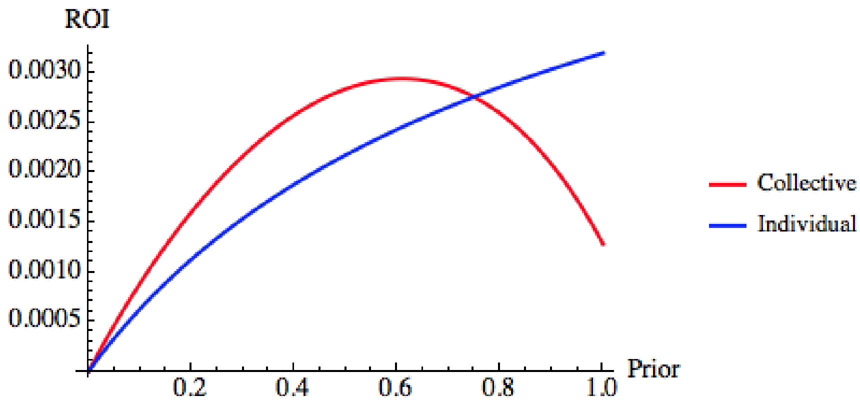


Figure 9: Comparison of Returns on Investment with $\pi_H = 1$
$\pi_L = 0.6, \ \delta = 0.2$

3.4 Interpretation of Results

So far, we have only stated the formal results and the intuition for why those results hold. Here, we summarize the economic implications and interpretation of the results in the context of country of origin.

First, the paramater μ can be thought of as a "base reputation" of a country. The question we would like to investigate here is: In a country with high μ, which industries have an incentive to make use of country of origin advertising. Which industries should advertise country of origin for countries with low μ?

By Propositions 1 and 2 for high μ, industries with exclusive knowledge, such as French wine, Swiss watches, German automobiles, Japanese electronics, US software, etc. can benefit from advertising country of origin. In contrast, producers of generic products such as screws, basic clothes, etc., are better off advertising their own brand only. In countries with low base reputation μ, those generic manufacturers can instead benefit from country of origin advertising. Producers of products that require exclusive knowledge are instead better off building their own brand. These interpretations are summarized in Table 3.4

These predictions are consistent with anecdotal evidence. For example the collective brand "Made in China" is advertised by subsuppliers on platforms such as 'Made-in-China.com‘,"

	Exclusive knowledge ($\pi_H \approx 1$)	Quality control ($\pi_L \approx 0$)
High base reputation ($\mu \approx 1$)	collective reputation has commitment value	individual reputation always better
Low base reputation ($\mu \approx 0$)	individual reputation always better	collective reputation has commitment value

Table 1: Summary of results

while successful high-tech companies such as Huawei rather try to build their own brand names. Instead German subsuppliers such as ThyssenKrupp rather count on their own brand reputation. Our results could also be applied to the labeling of "Made in Germany" versus "Made in Europe".

There are two ways to think about theses observations. First, one can argue that companies that pick the correct branding strategies will be the ones that survive and thus on average we should observe a selection of companies that apply the correct strategy. Another interpretation is that firms are actually choosing the best strategy taking into account the commitment value of country of origin.

Finally, these insights can play a role for the regulation of labeling of country of origin. While the classic argument is that companies should be required to label their product with certain information for consumer protection, imposing what aspects are emphasized and what implications this has on the brand and customer beliefs and moral hazard problem of firms involved.

4 General Analysis with $T-$Period Memory

In this section, we extend the model to a memory of an arbitrarily finite periods and verify that our results are robust. Intuitions for the results do not depend on the two-period assumption. In fact, we expect our results to be stronger with a longer memory. We saw in the main analysis that an individual brand is better at reaching a high or low level of reputation at which point it faced a low incentive for investing in quality. An individual brand that has built a very good or bad reputation is discouraged from investing further because it cannot improve consumers' beliefs sufficiently. Consumers' longer memory would worsen this problem as it will allow firms to be in a more extreme level of reputation after a longer streak of good or bad outcomes.

Now a relevant history is an element of $\mathcal{H}^{\text{ind}} := \{G, \emptyset, B\}^T$ for an individual brand and $\mathcal{H}^{\text{col}} := \{G, B\}^T$ for a collective brand. Suppose the focal investment decision is made at $t = 0$, and denote an outcome produced by an investment decision at any period t by $h_t \in \{G, \emptyset, B\}$. Then, the T-period history is a vector $\mathbf{h}_{old} = (h_{-T}h_{-T+1}...h_{-2}h_{-1})$. As the firm continues to make a sequence of investment decisions after $t = 0$, outcomes $h_1, h_2, ...$ are realized. We find it useful to denote a vector of consecutive outcomes between $t = i$ and $t = j$ by $\mathbf{h}^{i:j}$. So, for example $\mathbf{h}_{old} = \mathbf{h}^{-T:-1}$. Otherwise, we use notations that are natural extensions from those of the two-period model.

4.1 Individual reputation

The equilibrium analysis is analogous to that for the two-period model. The reputational equilibrium exists if and only if investing is an optimal decision after all possible histories. Equivalently, the benefit from investment should be greater than its cost for every history. Given a history $\mathbf{h}_{old}$, the benefit of the focal investment decision is realized through the next T periods until consumers forget h_0, or its outcome. By investing in quality, the firm can improve the outcome produced, thereby improving consumers' willingness to pay. Thus, the expected benefit is a present-discounted sum of price premium over T periods. We now state a lemma that characterizes the cutoff level for individual reputation.

Lemma 5. *For an individual firm, there exists a constant $\hat{c}^{ind} > 0$ such that a* **RE** *exists if and only if $c \leq \hat{c}^{ind}$ where*

$$\hat{c}^{ind} = \frac{\delta \Delta \pi}{2} \cdot \min_{\mathbf{h}_{old} \in \{G, \emptyset, B\}^T} \sum_{k=0}^{T-1} \delta^k \sum_{\mathbf{h}^{1:k} \in \{G, \emptyset, B\}^k} Pr(\mathbf{h}^{1:k}) (p(\mathbf{h}^{-T+k+1:-1} G \mathbf{h}^{1:k}) - p(\mathbf{h}^{-T+k+1:-1} B \mathbf{h}^{1:k})) \text{ .}^5 \tag{9}$$

[5] When $k = 0$, $\mathbf{h}^{1:k}$ is defined to be an emptyset. Therefore, $\mathbf{h}^{-T+k+1:-1} G \mathbf{h}^{1:k}$ is equivalent to $\mathbf{h}^{-T+1:1} G$.

This lemma is essentially a general version of lemma 1. As time proceeds to $t = k$, where $1 \leq k \leq T-1$, of the elements of the history $\mathbf{h}_{old}$, old ones are forgotten ($\mathbf{h}^{-T+k+1:-1}$), while new ones enter the memory ($\mathbf{h}^{1:k}$).

The source of the price premium is the differential element G and B in between older and newer outcomes. By investing in the focal period $t = 0$, the firm manages to have one more good outcome in the history observed by consumers. Therefore, the premium realized in $t = k$ conditional on the sequence of new outcomes $\mathbf{h}^{1:k}$ is $p(\mathbf{h}^{-T+k+1:-1}G\mathbf{h}^{1:k}) - p(\mathbf{h}^{-T+k+1:-1}B\mathbf{h}^{1:k})$.

In the short-run, most of the original history remains in consumers' memory ($\mathbf{h}^{-T+k+1:-1}$). Therefore, if a firm has built a very high level of reputation with many good outcomes in the past, the firm does not gain much through premiums realized in the near future by investing in $t = 0$. Similarly, a firm that has a very bad reputation at $t = 0$ has a low short-run incentive to invest.

What may motivate such firms with extreme reputation is the long-run incentive. A firm with good reputation may fear that a decision not to invest today may lead to a bad future reputation, especially if outcomes of its future investment ($\mathbf{h}^{1:k}$) is likely to be bad. On the other hand, if the firm expects them to be good, then the long-run incentive would be too low to discipline the firm's action today.

Recall that we are trying to compute the exact cutoff level $\hat{c}^{\text{ind}}$. This requires of knowing the history that provides the minimum benefit for given parameters. For this purpose, we consider again two special signal structures: exclusive technological know-how ($\pi_L = 0$) and quality control ($\pi_H = 1$). The former provides an environment where building an extremely high level of reputation is easy for a competent firm, as one good outcome completely reveals its type. Therefore, we can attain a small benefit from investment by choosing a history that has a lasting damage to the firm's incentives. This implies that any history $\mathbf{h}^{-T:1}$ with $h_{-1} = G$ does the job. Since the most recent outcome is good, consumers know perfectly the

Also, when $k = T-1$, $\mathbf{h}^{-T+k+1:-1}$ is equivalent to an emptyset so that $\mathbf{h}^{-T+k+1:-1}G\mathbf{h}^{1:k}$ is $G\mathbf{h}^{1:T-1}$.

firm's type to be good until $t = T - 2$. This eliminates any incentive for the firm to invest at $t = 0$ other than the incentive realized at the very last period $t = T - 1$. In other words, $p(\mathbf{h}^{-T+k+1:-1}G\mathbf{h}^{1:k}) - p(\mathbf{h}^{-T+k+1:-1}B\mathbf{h}^{1:k}) = 0$ for all $0 \leq k \leq T - 2$.

Under the structure of quality control, one bad outcome completely reveals a firm to be an incompetent type. Since the incentives are lower on the lower end of the reputation ladder, we find a history that has a lasting negative influence on the history observed by consumers. Then, $h_{-1} = B$ does the job. The most recent outcome reveals the firm to be incompetent to consumers from $t = 0$ to $t = T - 2$.

The following lemma summarizes analyses on the characterization of the cutoff-levels under two special signal structures.

Lemma 6. *1. Under the environment of exclusive technology, a competent firm expects the lowest benefit from an investment immediately following a good outcome. In this case, all of the firm's investment incentives vanish, except for the one realized in the longest run,* $t = T - 1$, *and the benefit is equivalent to:*

$$\lim_{\pi_L \to 0} \hat{c} = \frac{\delta^T \cdot \pi_H^2 (1-\mu)}{2^T} \cdot \left({}_{k=0}^{T-1} \quad \genfrac{}{}{0pt}{}{T-1}{k} \cdot \frac{(1-\pi_H)^k}{\mu(1-\pi_H)^{k+1} + (1-\mu)} \right). \tag{10}$$

2. Under the environment of quality control, an investment decision immediately following a bad outcome provides the lowest benefit and it is equivalent to:

$$\lim_{\pi_H \to 1} \hat{c}^{ind} = \frac{\delta^T (1-\pi_L)^2 \mu}{2^T} \cdot \left({}_{k=0}^{T-1} \quad \genfrac{}{}{0pt}{}{T-1}{k} \quad \frac{1}{\mu + (1-\mu)\pi_L^{k+1}} \right). \tag{11}$$

4.2 Collective reputation

A longer memory may allow a collective brand to achieve either a high or low level of reputation and cause a commitment problem to firms. However, as we saw in the analysis of the two-period model, consumers' limited observability for a collective brand alleviates this problem; As consumers cannot observe history at firm-level, they can never learn perfectly about the types of two firms in the group. Therefore, a competent firm can always improve the brand reputation by investing in quality.

The next lemma establishes the necessary and sufficient condition for the existence of reputational equilibrium. Let $\Pr(\mathbf{h}^{1:k};\theta)$ for $\mathbf{h}^{1:k} \in \{G,B\}^k$ and $\theta \in \{C,I\}$ be the probability that the brand of type θ produces a sequence of outcome $\mathbf{h}^{1:k}$ in k periods if a competent firm always invests.

Lemma 7. *For a competent firm within a collective brand, there exists a constant $\hat{c} > 0$ such that a* **RE** *exists if and only if $c \leq \hat{c}$ where*

$$\hat{c} = \frac{\delta\Delta\pi}{2} \cdot \min_{\mathbf{h}_{old},\theta} \sum_{k=0}^{T-1} \delta^k \sum_{\mathbf{h}^{1:k}\in\{G,B\}^k} Pr(\mathbf{f};\theta)(p(\mathbf{h}^{-T+k+1:-1}G\mathbf{h}^{1:k}) - p(\mathbf{h}^{-T+k+1:-1}B\mathbf{h}^{1:k})) \;, \quad (12)$$

where $\mathbf{h}_{old} \in \{G,B\}^T$ and $\theta \in \{C,I\}$.

This lemma is a general version of lemma 3. And the distinction we drew in the two-period model between the individual and collective reputation clearly applies here; the benefit from investment for a firm within a collective reputation depends on the type of the other firm.

Like in the individual reputation, a competent firm under a collective reputation has an incentive to invest in the focal period $t = 0$ because its good outcome will allow a better reputation of the brand in the next T periods, during which consumers will pay a premium. However, unlike in the case of the individual reputation, consumers' belief about the group quality always remains uncertain, even under special signal structures with $\pi_L = 0$ or $\pi_H = 1$. Technically, it also implies that the cutoff level $\hat{c}$ is very hard to compute, as none of the terms for the price premium vanishes. So, instead of computing the exact cutoff, we compute its lower bound and show that it is still greater than the exact cutoff for an individual brand, $\hat{c}^{\text{ind}}$.

First, suppose $\pi_L = 0$. Before choosing a lower bound, we need to identify the history that minimizes the benefit from investment and the type of the other firm. Since this is an environment that a brand can build up reputation relatively easily with a good outcome, the best possible history with all good outcomes $\mathbf{h}_{old} = G^T$ must provide the lowest benefit. Also, new outcomes that replace older memories after some periods are likely to maintain the clean sheet if the other firm is also competent. It in fact straightforward to show that for a large enough μ, $\mathbf{h}_{old} = G^T$ and $\theta = C$ together provide $\hat{c}$.

Now to find a lower bound for $\hat{c}$, we sum over a subset of all premiums. Note in equation 12 that the expected price premium in period $t = k$ depends on the realization of future outcomes, $\mathbf{h}^{1:k}$. We will focus on events that only good outcomes are realized, and treat other expressions as zero. Such an event occurs with a probability π_H^k since both firms of the group are competent. Conditional on this event, the price premium enjoyed by the firm is $p(G^T) - p(BG^{T-1})$. The one bad outcome in the latter term is due to the decision not to invest in period $t = 0$. Otherwise, the firm keeps the clean sheet under this event.

Second, suppose $\pi_H = 1$. This is an environment where a brand's reputation can fall as a consequence of bad outcomes. Therefore, a firm of a collective brand is most discouraged from investing when it has produced many bad outcomes. It is also straightforward to show that $\mathbf{h}_{old} = B^T$ and $\theta = I$ together provides $\hat{c}$ for small enough μ.

Similar to the previous case, we focus on the event that the brand only produces bad outcomes in the future, which happens with probability $(1 - \frac{\pi_H + \pi_L}{2})^k$ as there is one of each type in the firm. Conditional on this event, the firm's history would be GB^{T-1} if invested in $t = 0$, and B^T if not. Therefore, the expected price premium in this event in period $t = k$ is $p(GB^{T-1}) - p(B^T)$.

We summarize the analysis on collective reputation and the characterization of lower bounds in the next lemma:

Lemma 8. *1. Under the environment of quality control ($\pi_L = 0$), if μ is large enough, a competent firm of a collective brand faces the lowest benefit from investment if it has produced all good outcomes in the remembered history ($\mathbf{h}_{old} = G^T$) and the other firm is also competent ($\theta = C$). A lower bound for the cutoff level, $\underline{\hat{c}}_{(\pi_L=0)}$, is obtained by focusing on the event in which only good outcomes are realized ($\mathbf{h}^{1:k} = G^k$).*

for the other limit, and we may have more equilibria that coexist in that region.

Therefore, with our analysis thus far, we can state the following result.

Proposition 4. *Under exclusive technology ($\pi_L = 0$) and consumers' prior belief (μ) is close to 1, then a competent firm gains a greater profits by building a collective brand with an incompetent firm than that as an stand-alone firm.*

Proposition 4 shows that under exclusive technology, the commitment power that a firm gains through branding collectively can be large enough that it may find it optimal to group with an incompetent firm. Since the second best equilibrium for an individual brand is the worst equilibrium in which the firm never invests, the firm would gain substantially by forming a collective brand.

It remains to show that a similar result holds for the quality control case.

6 Conclusion

In this paper, we have examined models of collective and individual reputation. In particular, we have found that a collective brand sustains the good equilibrium better if either π_L is low and μ is high, or π_H is high and μ is low. We also explored a firm's endogenous branding decision and found a competent firm sometimes obtains a greater profit by grouping with an incompetent firm than it would alone. These two results together highlight the benefit of collective reputation. For firms facing a moral hazard problem, where competent firms alone cannot commit to invest always, collective brands may provide an additional commitment power to investment. And the commitment allows trust between firms and consumers and greater profits for firms to arise.

While our main contribution is in providing an explanation for why collective brands may overcome free-riding incentives and sustain good reputation, it is important to map our conditions for results onto practical examples. The prior μ would be largely determined by ex-ante beliefs about the quality across markets, industries and economies. And signal

A Appendix: Proofs

Proof. [Proof of Lemma 1] The posterior beliefs about the quality of the product after observing history h are given by

$$\hat{\mu}^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}, \qquad \hat{\mu}^{\text{ind}}(GB) = \hat{\mu}^{\text{ind}}(BG) = \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi^H) + (1-\mu)\pi_L(1-\pi^L)},$$

$$\hat{\mu}^{\text{ind}}(BB) = \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}, \qquad \hat{\mu}^{\text{ind}}(G\emptyset) = \hat{\mu}(\emptyset G) = \frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L}$$

$$\hat{\mu}^{\text{ind}}(\emptyset\emptyset) = \mu, \qquad \hat{\mu}(B\emptyset) = \hat{\mu}(\emptyset B) = \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}.$$

Investment by the competent firm can only be sustained if her incentive compatibility constraint is satisfied after every possible history. The firm invests after a history $(h,1)$ if and only if $V^{\text{ind}}(h,1) \geq \tilde{V}^{\text{ind}}(h,1)$ which is equivalent to

$$c \leq \hat{c}^{\text{ind}}(h_1) \equiv \frac{\delta(\pi_H - \pi_L)}{2} \cdot \underbrace{V^{\text{ind}}(h_1G,1) - V^{\text{ind}}(h_1B,1) + V^{\text{ind}}(h_1G,\emptyset) - V^{\text{ind}}(h_1B,\emptyset)}_{\equiv \hat{d}^{\text{ind}}(h_1)}.$$

Note that $\hat{d}^{\text{ind}}(h_1)$ can potentially depend on c. Using $V(h,\emptyset) = \frac{\delta}{2}\left(V(h_1\emptyset,1) + V(h_1\emptyset,\emptyset)\right)$ we can calculate

$$\begin{aligned} V^{\text{ind}}(h_{-1}G,1) - V^{\text{ind}}(h_{-1}B,1) &= p^{\text{ind}}(h_{-1}G) - p^{\text{ind}}(h_{-1}B) \\ &+ \frac{\delta}{2}\pi_H(\underbrace{V^{\text{ind}}(GG,1) - V^{\text{ind}}(BG,1)}_{=p^{\text{ind}}(GG)-p^{\text{ind}}(BG)} + \underbrace{V^{\text{ind}}(GG,\emptyset) - V^{\text{ind}}(BG,\emptyset)}_{=0}) \\ &+ \frac{\delta}{2}(1-\pi_H)(\underbrace{V^{\text{ind}}(GB,1) - V^{\text{ind}}(BB,1)}_{=p^{\text{ind}}(GB)-p^{\text{ind}}(BB)} + \underbrace{V^{\text{ind}}(GB,\emptyset) - V^{\text{ind}}(BB,\emptyset)}_{=0}). \end{aligned}$$

Similarly, $V^{\text{ind}}(h_{-1}G,\emptyset) - V^{\text{ind}}(h_{-1}B,\emptyset) = \frac{\delta}{2}(p^{\text{ind}}(G\emptyset) - p^{\text{ind}}(B\emptyset))$. □

Proof. [Proof of Lemma 2] First, note that

$$\begin{aligned}
p^{\text{ind}}(GG) - p^{\text{ind}}(GB) &= (\pi_H - \pi_L)(\frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2} - \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)}) \\
&= \mu\pi_H(\pi_H - \pi_L)(\frac{\pi_H}{\mu\pi_H^2 + (1-\mu)\pi_L^2} - \frac{1-\pi_H}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)}) \\
&= \frac{\mu(1-\mu)\pi_H\pi_L(\pi_H - \pi_L)^2}{\Pr(GG)\cdot\Pr(GB)},
\end{aligned}$$

and

$$\begin{aligned}
p^{\text{ind}}(GB) - p^{\text{ind}}(BB) &= (\pi_H - \pi_L)(\frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} - \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}) \\
&= \mu(1-\pi_H)(\pi_H - \pi_L)(\frac{\pi_H}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} - \frac{1-\pi_H}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}) \\
&= \frac{\mu(1-\mu)(1-\pi_H)(1-\pi_L)(\pi_H - \pi_L)^2}{\Pr(GB)\cdot\Pr(BB)},
\end{aligned}$$

and

$$\begin{aligned}
p^{\text{ind}}(G\emptyset) - p^{\text{ind}}(B\emptyset) &= (\pi_H - \pi_L)(\frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L} - \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}) \\
&= \frac{\mu(1-\mu)(\pi_H - \pi_L)^2}{\Pr(G)\cdot\Pr(B)} \geq \min\{p^{\text{ind}}(GG) - p^{\text{ind}}(GB),\ p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\}.
\end{aligned}$$

Hence, history $h_{-1} = G$ provides the binding constraint if and only if $\frac{\pi_H\pi_L}{\Pr(GG)\cdot\Pr(GB)} \leq \frac{(1-\pi_H)(1-\pi_L)}{\Pr(GB)\cdot\Pr(BB)}$, which holds if and only if

$$\begin{aligned}
&\Pr(BB)\cdot\pi_H\pi_L \leq \Pr(GG)\cdot(1-\pi_H)(1-\pi_L) \\
\Leftrightarrow\ &\pi_H\pi_L(\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2) \leq (1-\pi_H)(1-\pi_L)(\mu\pi_H^2 + (1-\mu)\pi_L^2) \\
\Leftrightarrow\ &\mu\pi_H(1-\pi_H) \geq (1-\mu)\pi_L(1-\pi_L)
\end{aligned}$$

This inequality holds if and only if $\mu \geq \bar{\mu} \equiv \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H)+\pi_L(1-\pi_L)}$. □

Proof. [Proof of Lemma 3] The **RE** exists if and only if $V(h,d;\theta) \geq \tilde{V}(h,d;\theta)$. A competent firm invests after a history $(h, 1)$ if and only if

$$c \leq \hat{c}(h_{-1}) \equiv \delta \cdot \frac{\pi_H - \pi_L}{2} \cdot \underbrace{\left(V(h_{-1}G, 1; \theta) - V(h_{-1}B, 1; \theta) + V(h_{-1}G, \emptyset; \theta) - V(h_{-1}B, \emptyset; \theta)\right)}_{\equiv \hat{d}(h_{-1};\theta)}.$$

First, note that for all $q_1, q_2, x \in \{G, B\}$, we have that $V(q_1x, 1) - V(q_2x, 1) = p(q_1x) - p(q_2x)$ and $V(q_1x, \emptyset; \theta) - V(q_2x, \emptyset; \theta) = 0$. Using this, we can calculate

$$\begin{aligned}
V(h_{-1}G, 1; \theta) - V(h_{-1}B, 1; \theta) &= p(h_{-1}G) - p(h_{-1}B) \\
&+ \frac{\delta\pi_H}{2}(V(GG, 1; \theta) - V(BG, 1; \theta)) + \frac{\delta(1 - \pi_H)}{2}(V(GB, 1; \theta) - V(BB, 1; \theta)) \\
&+ \frac{\delta\pi_H}{2}(V(GG, 0; \theta) - V(BG, 0; \theta)) + \frac{\delta(1 - \pi_H)}{2}(V(GB, 0; \theta) - V(BB, 0; \theta)) \\
&= p(h_{-1}G) - p(h_{-1}B) + \frac{\delta\pi_H}{2}(p(GG) - p(BG)) + \frac{\delta(1 - \pi_H)}{2}(p(GB) - p(BB))
\end{aligned}$$

Likewise,

$$\begin{aligned}
V(h_{-1}G, \emptyset; \theta) - V(h_{-1}B, \emptyset; \theta) =& \frac{\delta\pi(\theta)}{2}(V(GG, 1; \theta) - V(BG, 1; \theta)) + \frac{\delta(1 - \pi(\theta))}{2}(V(GB, 1; \theta) - V(BB, 1; \theta)) \\
&+ \frac{\delta\pi(\theta)}{2}(V(GG, 0; \theta) - V(BG, 0; \theta)) + \frac{\delta(1 - \pi(\theta))}{2}(V(GB, 0; \theta) - V(BB, 0; \theta)) \\
=& \frac{\delta\pi(\theta)}{2}(p(GG) - p(BG)) + \frac{\delta(1 - \pi(\theta))}{2}(p(GB) - p(BB))
\end{aligned}$$

where $\pi(\theta) = \pi_L$ if $\theta = I$ and π_H if $\theta = C$. □

Proof. [Proof of Lemma 4] First note that

$$\begin{aligned}
\hat{\eta}_{CC}(GG) &= \frac{\mu^2\pi_H^2}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\hat{\eta}_{CI}(GG) &= \hat{\eta}_{IC}(GG) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right)}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\
\hat{\eta}_{II}(GG) &= 1 - \hat{\mu}_{CC}(GG) - 2\hat{\eta}_{CI}(GG), \\
\hat{\eta}_{CC}(GB) &= \frac{\mu^2\pi_H(1-\pi_H)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\hat{\eta}_{CI}(GB) &= \hat{\eta}_{IC}(GB) \\
&= \frac{\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\
\hat{\eta}_{II}(GB) &= 1 - \hat{\eta}_{CC}(GB) - 2\hat{\eta}_{CI}(GB), \\
\hat{\eta}_{CC}(BB) &= \frac{\mu^2(1-\pi_H)^2}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\hat{\eta}_{CI}(BB) &= \hat{\eta}_{IC}(BB) \\
&= \frac{\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right)}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\
\hat{\eta}_{II}(BB) &= 1 - \hat{\eta}_{CC}(BB) - 2\hat{\eta}_{CI}(BB).
\end{aligned}$$

$$\begin{aligned}
p(GG) - p(GB) &= (\pi_H - \pi_L)\left(\frac{\mu^2\pi_H^2 + \frac{1}{4}\mu(1-\mu)(\pi_H+\pi_L)^2}{\Pr(GG)} - \frac{\mu^2\pi_H(1-\pi_H) + \frac{1}{4}\mu(1-\mu)(\pi_H+\pi_L)(2-\pi_H-\pi_L)}{\Pr(GB)}\right) \\
&= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)\right)}{4\cdot\Pr(GG)\cdot\Pr(GB)} \\
&\to_{\pi_L\to 0} \frac{(1-\mu)\mu\pi_H}{(1+\mu)(1-\pi_H+1-\mu\pi_H)} \\
p(GB) - p(BB) &= (\pi_H - \pi_L)\left(\frac{\mu^2\pi_H(1-\pi_H) + \frac{1}{4}\mu(1-\mu)(\pi_H+\pi_L)(2-\pi_H-\pi_L)}{\Pr(GB)} - \frac{\mu^2(1-\pi_H)^2 + \frac{1}{4}\mu(1-\mu)\left((1-\pi_H)+(1-\pi_L)\right)^2}{\Pr(BB)}\right) \\
&= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)\right)}{4\cdot\Pr(GB)\cdot\Pr(BB)} \\
&\to_{\pi_L\to 0} \frac{(1-\mu)\pi_H\left((1-\mu\pi_H)^2 + 1 - \pi_H\right)}{\left((1-\mu\pi_H)^2 + \mu(1-\pi_H)^2 + 1 - \mu\right)(1-\pi_H+1-\mu\pi_H)}
\end{aligned}$$

$p(GG) - p(GB) \leq p(GB) - p(BB)$ if and only if

$$\begin{aligned}
\frac{\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)}{\Pr(GG)} &\leq \frac{\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)}{\Pr(BB)} \\
&= \frac{\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L) - (2\mu-1)(\pi_H-\pi_L) + 2(1-\pi_H-\pi_L)}{\Pr(GG) - \mu(\pi_H-\pi_L)\pi_L + \frac{1}{2}}
\end{aligned}$$

Therefore, above condition can be re-written as $\frac{A}{C} \leq \frac{A+B}{C+D}$, where $A = \mu^2(\pi_H - \pi_L)^2 + 2\mu(\pi_H - \pi_L)\pi_L + \pi_L(\pi_H + \pi_L)$, $B = -(2\mu - 1)(\pi_H - \pi_L) + 2(1 - \pi_H - \pi_L)$, $C = \Pr(GG)$, and $D =$

$-\mu(\pi_H - \pi_L)\pi_L + \frac{1}{2}$. This holds if and only if $AD \leq BC \Leftrightarrow$

$$(\pi_H - \pi_L) \;\; 2\mu^3(\pi_H - \pi_L)^2 - 3\mu^2(\pi_H - \pi_L)^2 + \mu(2 - \pi_H - \pi_L)(\pi_H + \pi_L) - 2(1 - \pi_L)\pi_L \;\; \geq 0$$

$$f(\mu, \pi_H, \pi_L) \triangleq \mu^2(2\mu - 3)(\pi_H - \pi_L)^2 + \mu(2 - \pi_H - \pi_L)(\pi_H + \pi_L) - 2(1 - \pi_L)\pi_L \geq 0.$$

Note that if $\mu = 1$, the LHS is equivalent to $2(1 - \pi_H)\pi_H \geq 0$ and if $\mu = 0$, $-2(1 - \pi_L)\pi_L \leq 0$. Therefore, it vanishes at least once for some value of μ between 0 and 1. The question is whether it can vanishes more than once. To see when f is increasing in μ

$$\frac{\partial f}{\partial \mu} = \mu(\mu - 1) + \frac{(\pi_H + \pi_L)(2 - \pi_H - \pi_L)}{6(\pi_H - \pi_L)^2} > 0 \Leftrightarrow (1 - \mu)\mu < \frac{(\pi_H + \pi_L)(2 - \pi_H - \pi_L)}{6(\pi_H - \pi_L)^2}$$

First, if $\pi_H - \pi_L$ is small, RHS becomes large and the condition holds always, so f crosses 0 at a single point. It is when $\pi_H - \pi_L$ is substantially large that the condition holds for small and large values of μ. Then, though $f(0, \pi_H, \pi_L) < 0$, it increases in μ for μ close to 0, at which point f may cross 0 for the first time. Then, for intermediate values of μ, f decreases and may cross 0 one more time. Then, lastly f increases for large values of μ and cross 0 again.

Although we do not fully identify necessary and sufficient conditions for $f \geq 0$, under symmetric signals with $\pi_L = 1 - \pi_H < \frac{1}{2}$, $f(\mu, \pi_H, 1 - \pi_H) \geq 0$ if and only if $\frac{1}{2} < \pi_H \leq \frac{3+\sqrt{6}}{6}$, $\frac{1}{2} \leq \mu \leq 1$ or $\frac{3+\sqrt{6}}{6} < \pi_H < 1$ and $\mu \in [\frac{1}{2} - \frac{\sqrt{1-12\pi_H+12\pi_H^2}}{2}, \frac{1}{2}] \cup [\frac{1}{2} + \frac{\sqrt{1-12\pi_H+12\pi_H^2}}{2}, 1]$. This exactly coincides with the patter described above. □

Proof. [Proof of Proposition 1] First, not that it follows from Lemma 2 that in the limit ($\pi_L = 0$), $\hat{c}^{\text{ind}} = \hat{c}^{\text{ind}}(G)$ for all parameters. Similarly, it follows from Lemma 4 that $\hat{c} = \hat{c}(G; C)$ for $\pi_L = 0$ and high and low values of μ.

Moreover, under individual reputation,

$$\begin{aligned} \lim_{\pi_L \to 0} \hat{d^{\text{ind}}}(G) &= \lim_{\pi_L \to 0} (1 + \frac{\delta \pi_H}{2})(\underbrace{p(GG) - p(GB)}_{\to 0}) + \frac{\delta(1 - \pi_H)}{2}(p(GB) - p(BB)) + \frac{\delta}{2}(p(G\emptyset) - p(B\emptyset)) \\ &= \delta \pi_H (1 - \mu) \quad \frac{(1 - \pi_H)}{2} \frac{1}{1 - \mu\pi_H(2 - \pi_H)} + \frac{1}{2} \frac{1}{1 - \mu\pi_H} \quad . \end{aligned}$$

For the case of collective reputation,

$$\begin{aligned}\lim_{\pi_L\to 0}\hat{d}(G;C) &= \lim_{\pi_L\to 0}(1+\frac{\delta\pi_H}{2})(p(GG)-p(GB))+\frac{\delta(1-\pi_H)}{2}(p(GB)-p(BB))\\ &= (1-\mu)\pi_H\cdot\Bigg[\Bigg(1+\frac{\delta\pi_H}{2}\quad\frac{(1-\mu)\mu\pi_H}{(1+\mu)(2-(1+\mu)\pi_H)}\\ &\quad+\frac{\delta(1-\pi_H)}{2}\frac{(1-\mu\pi_H)^2+1-\pi_H}{\left((1-\mu\pi_H)^2+\mu(1-\pi_H)^2+1-\mu\right)(2-(1+\mu)\pi_H)}\Bigg]\end{aligned}$$

First, to compare the two cutoffs for μ close to 1, note that

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{(1-\pi_H)}{2}\frac{1}{1-\mu\pi_H(2-\pi_H)}+\frac{1}{2}\frac{1}{1-\mu\pi_H}\quad=\quad\frac{1}{1-\pi_H}$$

and

$$\begin{aligned}&\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{(1+\frac{\delta\pi_H}{2})\mu}{(1+\mu)(2-(1+\mu)\pi_H)}+\frac{\delta(1-\pi_H)}{2}\frac{(1-\mu\pi_H)^2+1-\pi_H}{\left((1-\mu\pi_H)^2+\mu(1-\pi_H)^2+1-\mu\right)(2-(1+\mu)\pi_H)}\\ &\qquad=(1+\frac{\delta\pi_H}{2})\frac{1}{2(2-2\pi_H)}+\frac{\delta(1-\pi_H)}{2}\frac{(1-\pi_H)^2+1-\pi_H}{\left((1-\pi_H)^2+(1-\pi_H)^2\right)(2-2\pi_H)}\\ &\qquad=\frac{1+\delta}{4(1-\pi_H)}.\end{aligned}$$

Since $\frac{1+\delta}{4(1-\pi_H)}\geq\frac{\delta}{1-\pi_H}$ if and only if $\delta<\frac{1}{3}$, for sufficiently small μ collective brands are better whenever $\delta<\frac{1}{3}$.

Similarly, to compare the two cutoffs for μ close to 0, note that

$$\lim_{\mu\to 0}\lim_{\pi_L\to 0}(1-\mu)\frac{(1-\pi_H)}{2}\frac{1}{1-\mu\pi_H(2-\pi_H)}+\frac{1}{2}\frac{1}{1-\mu\pi_H}\quad=\quad\frac{2-\pi_H}{2}$$

and

$$\begin{aligned}&\lim_{\mu\to 0}\lim_{\pi_L\to 0}\frac{(1+\frac{\delta\pi_H}{2})\mu}{(1+\mu)(2-(1+\mu)\pi_H)}+\frac{\delta(1-\pi_H)}{2}\frac{(1-\mu\pi_H)^2+1-\pi_H}{\left((1-\mu\pi_H)^2+\mu(1-\pi_H)^2+1-\mu\right)(2-(1+\mu)\pi_H)}\\ &\qquad=\frac{\delta(1-\pi_H)}{4}.\end{aligned}$$

Since $\delta\frac{2-\pi_H}{2}\geq\frac{\delta(1-\pi_H)}{4}$ for all $\pi_H\in[0,1]$, individual reputation building is better for sufficiently low priors. □

Proof. [Proof of Proposition 2] The proof is analogous to the proof of Proposition 1. It follows from Lemma 2 that in the limit $\pi_H \to 1$, $\hat{c}^{\text{ind}} = \hat{c}^{\text{ind}}(B)$ for all parameters. Similarly, it follows from Lemma 4 that $\hat{c} = \hat{c}(B; I)$ for high and low values of μ as $\pi_H \to 1$.

Under individual reputation

$$\begin{aligned}\lim_{\pi_H \to 1} \hat{d}^{\text{ind}}(B) &= \lim_{\pi_H \to 1} \frac{\delta}{2}(p^{\text{ind}}(GG) - p^{\text{ind}}(GB)) + (p^{\text{ind}}(GB) - p^{\text{ind}}(BB)) + \frac{\delta}{2}(p^{\text{ind}}(G\emptyset) - p^{\text{ind}}(B\emptyset)) \\ &= \frac{\delta}{2} \quad \frac{\mu(1-\pi_L)}{\mu + (1-\mu)\pi_L^2} + \frac{\mu(1-\pi_L)}{\mu + (1-\mu)\pi_L}\end{aligned}$$

and under collective reputation

$$\begin{aligned}\lim_{\pi_H \to 1} \hat{d}(B; I) &= \lim_{\pi_H \to 1} (p(BG) - p(BB)) + \frac{\delta}{2} \cdot ((1+\pi_L) \cdot (p(GG) - p(GB)) + (1-\pi_L)(p(GB) - p(BB))) \\ &= \frac{\mu(1-\pi_L)}{2} \cdot \\ &\quad \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta + 2\pi_L + 3\delta\pi_L) + \mu(2 + \delta + 4(1+\delta)\pi_L - (10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L + \delta(-1+3\pi_L))}{(2-\mu)(\mu(1-\pi_L) + 2\pi_L)(\mu(1+\mu) + 2(1-\mu)\mu\pi_L + (2-\mu)(1-\mu)\pi_L^2)}\end{aligned}$$

To compare the two cutoffs for μ close to 0, note that

$$\begin{aligned}\lim_{\mu \to 0} \quad & \frac{\delta}{\mu + (1-\mu)\pi_L^2} + \frac{\delta}{\mu + (1-\mu)\pi_L} - \\ & \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta + 2\pi_L + 3\delta\pi_L) + \mu(2 + \delta + 4(1+\delta)\pi_L - (10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L + \delta(-1+3\pi_L))}{2(2-\mu)(\mu(1-\pi_L) + 2\pi_L)(\mu(1+\mu) + 2(1-\mu)\mu\pi_L + (2-\mu)(1-\mu)\pi_L^2)} \\ = \quad & \frac{\delta}{\pi_L^2} + \frac{\delta}{\pi_L} - \frac{\delta + 2\pi_L + 3\delta\pi_L}{4\pi_L^2} > 0.\end{aligned}$$

Thus, for μ close to 0, $\hat{d}^{\text{ind}}(B) < \hat{d}(B; I)$ whenever $\delta < \frac{2\pi_L}{3+\pi_L}$.

To compare the two cutoffs for μ close to 1, note that

$$\begin{aligned}\lim_{\mu \to 1} \quad & \frac{1}{\mu + (1-\mu)\pi_L^2} + \frac{1}{\mu + (1-\mu)\pi_L} - \\ & \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta + 2\pi_L + 3\delta\pi_L) + \mu(2 + \delta + 4(1+\delta)\pi_L - (10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L + \delta(-1+3\pi_L))}{2(2-\mu)(\mu(1-\pi_L) + 2\pi_L)(\mu(1+\mu) + 2(1-\mu)\mu\pi_L + (2-\mu)(1-\mu)\pi_L^2)} \\ = \quad & 2 - \frac{1}{4}\delta(1+\pi_L) > 0.\end{aligned}$$

Thus, for μ close to 1, $\hat{d}^{\text{ind}}(B) > \hat{d}(B; I)$. □

Proof. [Proof of Proposition 4] To compute the stationary distribution over histories, define the probability the firm will produce a good signal after a history $h \in \mathcal{H}$ by $\pi^{\mathcal{S}}(h) = \sigma^{\mathcal{S}}(h) \cdot \pi_H + (1 -$

$\sigma^{\mathcal{S}}(h)) \cdot \pi_L$. Since the firm invests in quality if and only if $h \in \mathcal{S}$, $\pi^{\mathcal{S}}(h) = \pi_H$ if and only if $h \in \mathcal{S}$, and π_L otherwise.

$$
\begin{aligned}
\Pr_C^S(GG) &= \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))} \cdot \frac{\pi^{\mathcal{S}}(G)}{2},\ \Pr_C^S(GB) = \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))} \cdot \frac{1-\pi^{\mathcal{S}}(G)}{2} \\
\Pr_C^S(G\emptyset) &= \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))} \cdot \frac{1}{2},\ \Pr_C^S(BG) = \left(\frac{1}{2} - \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))}\right) \cdot \frac{\pi^{\mathcal{S}}(B)}{2} \\
\Pr_C^S(BB) &= \left(\frac{1}{2} - \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))}\right) \cdot \frac{1-\pi^{\mathcal{S}}(B)}{2},\ \Pr_C^S(B\emptyset) = \left(\frac{1}{2} - \frac{\pi^{\mathcal{S}}(B)+\pi^{\mathcal{S}}(\emptyset)}{2(2+\pi^{\mathcal{S}}(B)-\pi^{\mathcal{S}}(G))}\right) \cdot \frac{1}{2} \\
\Pr_C^S(\emptyset G) &= \frac{1}{2} \cdot \frac{\pi^{\mathcal{S}}(\emptyset)}{2},\ \Pr_C^S(\emptyset B) = \frac{1}{2} \cdot \frac{1-\pi^{\mathcal{S}}(\emptyset)}{2},\ \Pr_C^S(\emptyset\emptyset) = \frac{1}{2} \cdot \frac{1}{2},
\end{aligned}
$$

Since an incompetent firm cannot invest in quality, plugging in $\mathcal{S} = \emptyset$ to probabilitie above, $Pr_C^S(h)$, for all histories, we obtain

$$
\begin{aligned}
\Pr_I^S(GG) &= \frac{\pi_L^2}{4},\ \Pr_I^S(GB) = \Pr_I^S(BG) = \frac{\pi_L(1-\pi_L)}{4} \\
\Pr_I^S(G\emptyset) &= \Pr_I^S(\emptyset G) = \frac{\pi_L}{4},\ \Pr_I^S(B\emptyset) = \Pr_I^S(\emptyset B) = \frac{1-\pi_L}{4} \\
\Pr_I^S(BB) &= \frac{(1-\pi_L)^2}{4},\ \Pr_I^S(\emptyset\emptyset) = \frac{1}{2} \cdot \frac{1}{2}.
\end{aligned}
$$

□

Proof. [Proof of Lemma 5] So far, we have denoted a firm's value function of the firm by$V(\cdot)$, the present-discounted profit once the current customer visits the firm. We find it useful to introduce a notation for a value function prior to the customer's assignment, $W(\cdot)$, which we define:

$$
W(\mathbf{h}^T) \equiv \underbrace{\frac{1}{2}(p(\mathbf{h}^T)-c)}_{\text{expected current period}} + \delta \left(\underbrace{\frac{\pi_H}{2} \cdot W(\mathbf{h}^{T-1}G) + \frac{1-\pi_H}{2} \cdot W(\mathbf{h}^{T-1}B)}_{\text{future profit if visited this period}} + \underbrace{\frac{1}{2} \cdot W(\mathbf{h}^{T-1}\emptyset)}_{\text{otherwise}} \right).
$$

In the current period, the firm's expected profit is the probability of having the customer visit, $\frac{1}{2}$, multiplied by the profit margin, $p(\mathbf{h}^T) - c$. The firm can be under three different information sets, each of which generate different profits accordingly. With probability $\frac{\pi_H}{2}$ the firm is visited today and a produces a good history, bringing about a stream of profits summarized by $W(\mathbf{h}^{T-1}G)$. With probability $\frac{1-\pi_H}{2}$, the firm produces a bad history and generates a profit of $W(\mathbf{h}^{T-1}B)$. Lastly, with probability a half, the firm is not chosen today and therefore generates an empty signal. Note that $W(\cdot)$ is the firm's expected present-discounted payoff before the consumer's visit is determined.

If the firm is visited, the firm makes the investment decision by comparing expected payoffs from investment and no investment, denoted by V^* and $\hat{V}$, respectively.

$$V^*(\mathbf{h}^T) = p(\mathbf{h}^T) - c + \delta(\pi_H \cdot W(\mathbf{h}^{T-1}G) + (1-\pi_H) \cdot W(\mathbf{h}^{T-1}B)),$$
$$\hat{V}(\mathbf{h}^T) = p(\mathbf{h}^T) + \delta(\pi_L \cdot W(\mathbf{h}^{T-1}G) + (1-\pi_L) \cdot W(\mathbf{h}^{T-1}B)).$$

The **RE** exists iff $V^*(\mathbf{h}^T) \geq \hat{V}(\mathbf{h}^T)$, which holds if and only if

$$c \leq \hat{c}^{\text{ind}} \equiv \delta(\pi_H - \pi_L) \cdot \min_{\mathbf{h}^{T-1} \in \{G, \emptyset, B\}^{T-1}} \Delta W(\mathbf{h}^{T-1}), \tag{13}$$

where $\Delta W(\mathbf{h}^{T-1}) := W(\mathbf{h}^{T-1}G) - W(\mathbf{h}^{T-1}B)$ defines reputational benefit to be realized in the future generated by today's investment decision, conditional on the relevant history $\mathbf{h}^{T-1}$. The expression for $\hat{c}^{\text{ind}}$ is intuitive. The benefit from an investment in the current period comes from an increase in the future payoff, which is increasing in the discount factor δ and the informativeness of different signals, $\Delta\pi := \pi_H - \pi_L$. $\Delta W(\mathbf{h}^{T-1})$ summarizes the future reputational benefit through continued sum of price premiums that the decision to investment creates.

As in the analysis with two-period memory, we focus on parameter regions with small values of π_L by taking a limit $\pi_L \to 0$. Then, one good history fully reveals that the firm is competent. We now compute $\Delta W(\cdot)$ for a general model with a finite history length $t \geq 3$ and identify the binding constraint to characterize $\hat{c}^{\text{ind}}$. First,

$$
\begin{aligned}
W(\mathbf{h}^{T-1}G) &= \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{First } T-1 \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\Pr(\mathbf{g})(p(\mathbf{g})-c)}_{\text{After } T \text{ Periods}}. \\
&= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{\mathbf{G}(\mathbf{f})=i,\mathbf{B}(\mathbf{f})=j}p(\mathbf{h}^{T-1-k}G\mathbf{f})\right)-c\right) \\
&+\frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{\mathbf{G}(\mathbf{f})=i,\mathbf{B}(\mathbf{f})=j}p(\mathbf{f})\right)-c\right).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
W(\mathbf{h}^{T-1}B) &= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{\mathbf{G}(\mathbf{f})=i,\mathbf{B}(\mathbf{f})=j}p(\mathbf{h}^{T-1-k}B\mathbf{f})\right)-c\right) \\
&+\frac{1}{2}\delta^T\sum_{k=0}^{\infty}\delta^k\left(\sum_{i+j+l=T}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{\mathbf{G}(\mathbf{f})=i,\mathbf{B}(\mathbf{f})=j}p(\mathbf{f})\right)-c\right).
\end{aligned}
$$

Therefore, subtracting the two gives

$$
\Delta W(\mathbf{h}^{T-1}) = \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-p(\mathbf{h}^{T-k-1}B\mathbf{f})) \tag{14}
$$

$$
= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j+l=k}(\frac{\pi_H}{2})^i(\frac{1-\pi_H}{2})^j(\frac{1}{2})^l\left(\sum_{\mathbf{G}(\mathbf{f})=i,\mathbf{B}(\mathbf{f})=j}(p(\mathbf{h}^{T-1-k}G\mathbf{f})-p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\right) \tag{15}
$$

□

Proof. [Proof of Lemma **??**] We simplify the expression above by computing the price difference, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$. With $\pi_L \to 0$, $p(\mathbf{h}) = \pi_H\hat{\mu}(\mathbf{h})$ for any history $\mathbf{h}$. Therefore, we must find $\pi_H(\hat{\mu}(\mathbf{h}^{T-1-k}G\mathbf{f}) - \hat{\mu}(\mathbf{h}^{T-1-k}B\mathbf{f}))$ for any $\mathbf{f} \in \mathcal{H}^k$. Because a good signal fully reveals that a firm is competent, $\Delta\hat{\mu}(\mathbf{h}^{T-1}, k, \mathbf{f}) \equiv \hat{\mu}(\mathbf{h}^{T-1-k}G\mathbf{f}) - \hat{\mu}(\mathbf{h}^{T-1-k}B\mathbf{f}) = 0$ if and only if$\mathbf{G}(\mathbf{h}^{T-1-k}B\mathbf{f}) \geq 1$. Since our current goal is to find a history $\mathbf{h}^{T-1}$ that minimizes $\Delta W(\cdot)$, and given that $\Delta\hat{\mu}(\mathbf{h}^{T-1}, k, \mathbf{f}) \geq 0$ always, it is clear that we want $\Delta\hat{\mu} = 0$ for as many $\mathbf{f}$ as possible. Therefore we require $h_1 = G$, where h_1 is the most recent history of the firm's history at the time of the investment decision ($\mathbf{h}^T = h_T h_{T-1}...h_2 h_1$). Then, $\Delta\hat{\mu} = 0$ for all $\mathbf{f}$ for all $k = 0, 1, ..., T-2$. However, when $k = T-1$, T periods have passed after the investment, and the entire history $\mathbf{h}^T$ is forgotten and $\Delta\hat{\mu}(\emptyset, T-1, \mathbf{f}) = \hat{\mu}(G\mathbf{f}) - \hat{\mu}(B\mathbf{f})$, where $\mathbf{f} \in \mathcal{H}^{T-1}$. Again, this vanishes if and only if $\mathbf{G}(\mathbf{f}) \geq 1$. Therefore, $\Delta\hat{\mu}(\cdot)$ is positive if and only if T periods have passed, and none of the new history generated was good.

$$\lim_{\pi_L \to 0} \Delta W(\mathbf{h}^{T-1}) \quad = \quad \frac{\pi_H}{2} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1-\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \cdot \lim_{\pi_L \to 0} \hat{\mu}(GB^j\emptyset^{T-1-j}) - \hat{\mu}(B^{j+1}\emptyset^{T-1-j}) \right).$$

$\hat{\mu}(GB^j\emptyset^{T-1-j}) = 1$ because a good history causes a full revelation, and $\hat{\mu}(B^{j+1}\emptyset^{T-1-j}) = \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1}+1-\mu}$. Therefore,

$$\lim_{\pi_L \to 0} \Delta W(\mathbf{h}^{T-1}) = \frac{\pi_H(1-\mu)}{2^T} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right),$$

and consequently,

$$\lim_{\pi_L \to 0} \hat{c}^{\text{ind}} = \frac{\delta^T \pi_H^2 (1-\mu)}{2^T} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \cdot \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right).$$

□

Proof. [Proof of Lemma **??**]

$$\lim_{\pi_H \to 1} \Delta W(\mathbf{h}^{T-1}) \quad = \quad \lim_{\pi_H \to 1} \frac{\delta^{T-1}(\pi_H - \pi_L)}{2} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{\pi_H}{2})^j (\frac{1}{2})^{T-1-j} \ \hat{\mu}(G^{j+1}\emptyset^{T-1-j}) - \hat{\mu}(BG^j\emptyset^{T-1-j}) \right).$$

$\hat{\mu}(BG^j\emptyset^{T-1-j}) = 0$ because a good history causes a full revelation, and $\hat{\mu}(G^{j+1}\emptyset^{T-1-j}) = \frac{\mu}{\mu+(1-\mu)\pi_L^{j+1}}$. Therefore,

$$\lim_{\pi_H \to 1} \Delta W(\mathbf{h}^{T-1}) = \frac{\delta^{T-1}(1-\pi_L)\mu}{2^T} \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \right),$$

and consequently,

$$\lim_{\pi_H \to 1} \hat{c}^{\text{ind}} = \frac{\delta^T (1-\pi_L)^2 \mu}{2^T} \cdot \left(\sum_{j=0}^{T-1} \binom{T-1}{j} \frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \right).$$

□

Proof. [Proof of Lemma **??**] Let $\Delta\hat{\eta}(i) := \hat{\eta}(\mathbf{g}_2) - \hat{\eta}(\mathbf{g}_1)$ where $\mathbf{G}(\mathbf{g}_2) = \mathbf{G}(\mathbf{g}_1) + 1 = i + 1$. That is, $\Delta\hat{\eta}(i)$ denotes the difference in two posteriors where $\mathbf{g}_1$ has i good outcomes, one less than $\mathbf{g}_2$ does. Then,

$$
\begin{aligned}
& \sum_{\mathbf{f}\in\{G,B\}^k} \Pr(\mathbf{f},\theta) \cdot (\hat{\eta}(\mathbf{h}^{T-k-1}G\mathbf{f}) - \hat{\eta}(\mathbf{h}^{T-k-1}B\mathbf{f})) \\
\geq \; & \pi_H^k(\hat{\eta}(G^T) - \hat{\eta}(BG^{T-1})) \\
= \; & \pi_H^k \left(\frac{\mu\pi_H^T + (1-\mu)\cdot\left(\frac{\pi_H}{2}\right)^T}{\mu\pi_H^T + 2(1-\mu)\cdot\left(\frac{\pi_H}{2}\right)^T} - \frac{\mu\pi_H^{T-1}(1-\pi_H) + (1-\mu)\cdot\left(\frac{\pi_H}{2}\right)^{T-1}\left(1-\frac{\pi_H}{2}\right)}{\mu\pi_H^{T-1}(1-\pi_H) + 2(1-\mu)\cdot\left(\frac{\pi_H}{2}\right)^{T-1}\left(1-\frac{\pi_H}{2}\right)} \right) \\
= \; & \pi_H^k \left(\frac{A\pi_H + B\left(\frac{\pi_H}{2}\right)}{A\pi_H + 2B\left(\frac{\pi_H}{2}\right)} - \frac{A(1-\pi_H) + B\left(1-\frac{\pi_H}{2}\right)}{A(1-\pi_H) + 2B\left(1-\frac{\pi_H}{2}\right)} \right) \\
= \; & \pi_H^k \cdot \frac{AB\pi_H}{2\left(A\pi_H + 2B\left(\frac{\pi_H}{2}\right)\right)\left(A(1-\pi_H) + 2B\left(1-\frac{\pi_H}{2}\right)\right)} \\
= \; & \pi_H^k \cdot \frac{\mu(1-\mu)\frac{\pi_H^{2(T-1)}}{2^{T-1}}}{2\left(\mu\pi_H^{T-1} + (1-\mu)\left(\frac{\pi_H}{2}\right)^{T-1}\right)\left(\mu\pi_H^{T-1}(1-\pi_H) + 2(1-\mu)\left(\frac{\pi_H}{2}\right)^{T-1}\left(1-\frac{\pi_H}{2}\right)\right)} \\
= \; & \frac{\pi_H^k\mu(1-\mu)}{2^T\left(\mu + \frac{1-\mu}{2^{T-1}}\right)\left(\mu(1-\pi_H) + \frac{1-\mu}{2^{T-1}}(2-\pi_H)\right)}.
\end{aligned}
$$

where $A = \mu\pi_H^{T-1}$ and $B = (1-\mu)\left(\frac{\pi_H}{2}\right)^{T-1}$. Therefore, a lower bound for

$$
\Delta W(\mathbf{h}^{T-1}) \geq \underline{\Delta W(\mathbf{h}^{T-1})} := \frac{1-\delta^T\pi_H^T}{1-\delta\pi_H} \cdot \frac{\mu(1-\mu)\pi_H}{2^{T+1}\left(\mu + \frac{1-\mu}{2^{T-1}}\right)\left(\mu(1-\pi_H) + \frac{1-\mu}{2^{T-1}}(2-\pi_H)\right)},
$$

and $\hat{c} \geq \delta\pi_H \cdot \underline{\Delta W(\mathbf{h}^{T-1})}$. □

Proof. [Proof of Proposition **??**]

$$
\frac{1-\delta^T\pi_H^T}{1-\delta\pi_H} \cdot \frac{\delta\mu(1-\mu)\pi_H^2}{2^{T+1}\left(\mu + \frac{1-\mu}{2^{T-1}}\right)\left(\mu(1-\pi_H) + \frac{1-\mu}{2^{T-1}}(2-\pi_H)\right)} > \frac{\delta^T\pi_H^2(1-\mu)}{2^T}\sum_{j=0}^{T-1}\binom{T-1}{j} \cdot \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)},
$$

Now we plug $\pi_H \to 1$. RHS becomes $\frac{\delta^T}{2^T}$ and LHS $\frac{1-\delta^T}{1-\delta} \cdot \frac{2^{T-2}\delta\mu}{(2^T-2)\mu+2}$.

$$\frac{1-\delta^T}{1-\delta} \cdot \frac{2^{T-2}\delta\mu}{(2^T-2)\mu+2} > \frac{\delta^T}{2^T}$$
$$\frac{(1-\delta^T)2^{T-2}}{(1-\delta)\delta^{T-1}} > \frac{(2^T-2)\mu+2}{2^T\mu} = 1 + \frac{2(1-\mu)}{2^T\mu}.$$

LHS is decreasing in δ, while RHS is decreasing in μ. Since we are considering a large μ and $T \geq 3$, RHS is at most $1 + \frac{1}{4} = \frac{4}{5}$. RHS is equivalent to $2^{T-2}(1 + \frac{1}{\delta} + \cdots + \frac{1}{\delta^{T-1}}) > 2$. Therefore, for all values of δ and $T \geq 3$, the condition holds. □

Proof. [Proof of Proposition **??**] Now we find $\lim_{\pi_H \to 1}(\hat{\eta}(GB^{T-1}) - \hat{\eta}(B^T))$

$$\begin{aligned}
&= \frac{\mu \cdot \left(\frac{1+\pi_L}{2}\right)\left(\frac{1-\pi_L}{2}\right)^{T-1}}{2\mu \cdot \left(\frac{1+\pi_L}{2}\right)\left(\frac{1-\pi_L}{2}\right)^{T-1} + (1-\mu)\pi_L(1-\pi_L)^{T-1}} - \frac{\mu \cdot \left(\frac{1-\pi_L}{2}\right)^{T}}{2\mu \cdot \left(\frac{1-\pi_L}{2}\right)^{T} + (1-\mu)(1-\pi_L)^T} \\
&= \frac{\mu(1+\pi_L)}{2\mu(1+\pi_L) + 2^T(1-\mu)\pi_L} - \frac{\mu}{2\mu + 2^T(1-\mu)} \\
&= \frac{2^T\mu(1-\mu)}{(2\mu(1+\pi_L) + 2^T(1-\mu)\pi_L)(2\mu + 2^T(1-\mu))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\pi_H \to 1} \underline{\Delta W(B^{T-1}; I)} &= \left(\frac{1-\pi_L}{2}\right) \cdot \sum_{k=0}^{T-1} \delta^k \left(\frac{1-\pi_L}{2}\right)^k \frac{2^T\mu(1-\mu)}{(2\mu(1+\pi_L) + 2^T(1-\mu)\pi_L)(2\mu + 2^T(1-\mu))} \\
&= \left(\frac{1-\pi_L}{2}\right) \cdot \frac{1 - \left(\frac{(1-\pi_L)\delta}{2}\right)^T}{1 - \frac{(1-\pi_L)\delta}{2}} \cdot \frac{2^T\mu(1-\mu)}{(2\mu(1+\pi_L) + 2^T(1-\mu)\pi_L)(2\mu + 2^T(1-\mu))}.
\end{aligned}$$

Now, we compare $\lim_{\pi_H \to 1} \underline{c}(B^{T-1}; I)$ and $\lim_{\pi_H \to 1} \hat{c}^{\text{ind}}$ to find a sufficient condition for $\hat{c} > \hat{c}^{\text{ind}}$ for a large π_H. The comparison is still complicated, and we take the limit $\pi_L \to 0$. From equation 11, $\hat{c}^{\text{ind}}$ converges to δ^T, while $\underline{c}$ converges to $\frac{\delta}{4} \cdot \frac{1-(\frac{\delta}{2})^T}{1-\frac{\delta}{2}} \cdot \frac{2^T(1-\mu)}{(2\mu+2^T(1-\mu))}$. Then, $\lim_{\pi_L \to 0} \lim_{\pi_H \to 1} \underline{c}(B^{T-1}; I) > \lim_{\pi_L \to 0} \lim_{\pi_H \to 1} \hat{c}$ if and only if

$$\frac{\delta}{4} \cdot \frac{1-(\frac{\delta}{2})^T}{1-\frac{\delta}{2}} \cdot \frac{2^T(1-\mu)}{(2\mu+2^T(1-\mu))} \geq \delta^T$$
$$\Leftrightarrow \frac{1}{4} \cdot \frac{1-(\frac{\delta}{2})^T}{(1-\frac{\delta}{2})\delta^{T-1}} \geq 1 + \frac{2\mu}{2^T(1-\mu)}.$$

RHS is increasing in μ, while LHS is decreasing in δ. Since LHS is less than 1 if $\delta = 1$, which then is always less than RHS, the condition holds for δ not too large. $\square$