Sequential Innovation and Patent Policy

Álvaro Parra*

October 28, 2013

Abstract

I study how patent policy shapes R&D investments in the context of sequential innovation. I show that investments are driven by the incremental rent that firms obtain from innovating and increase with the approximation of the patent's expiration date. The main finding is that patent policy affects both the value of a new patent and the cost of replacing currently active patents. As a result, strong patent protection may decrease investment rates and, consequently, the economy's speed of innovation. In other words, the welfare losses of a protective policy lie beyond the loss in consumer surplus due to monopoly power.

1 Introduction

Inventions and the development of new knowledge are at the heart of the progress of modern economies. Much of the microeconomic dynamics within markets are generated by temporary competitive advantages created by the introduction of new products or the adoption of new production processes. Patents are the main tool that governments use to promote innovation. By granting the right of exclusion, patents help firms to obtain enough profits to incentivize the pursuit of new discoveries. However, the incentives provided by a patent system are not yet fully understood. The goal of this paper is to answer the following questions on the effects of patent policy in a context of *sequential* innovation (i.e. where innovations build upon each other): 1) what are the *dynamic* incentives induced by patent policy on firms' R&D decisions? 2) Given these incentives, what are the key determinants of an optimal patent policy and how does this policy relate to potentially observable market characteristics?

The classic work on patents (e.g. Nordhaus, 1969; Loury, 1979; and Lee and Wilde, 1980) treats each innovation as an isolated phenomenon, ignoring how the prospect of future breakthroughs affect the value of an innovation and, ultimately, the incentives to innovate at any given

*Álvaro Parra: Sauder School of Business, UBC, alvaro.parra@sauder.ubc.ca. I am indebted to Michael Whinston and Wojciech Olszewski for their advice and support. I also thank Nancy Gallini, Jose Espin-Sanchez, Thomas Hellmann, Jorge Lemus, Fernando Luco, Guillermo Marshall, Alessandro Pavan, Bruno Strulovici, Martin Szydlowski and seminar participants at Northwestern University, University of British Columbia, Washington University, IESE, Mannheim University, ESEM 2013, Bates and White. I gratefully acknowledge financial support of Northwestern's Center of Economic Theory.

point in time. This literature sustains the idea that, in the absence of economies of scale in R&D, longer and more enforceable (stronger) patents increase the returns of a succeeding innovator. Consequently, the design of an optimal patent system consists only in assessing the benefit of higher innovation rates, induced by stronger patents, and the monopoly cost associated with them. As we shall see, in a sequential world this logic is not complete. Even though patents are necessary to induce R&D, patent systems that are "too strong" induce a relatively low innovation arrival rate in the economy, implying that the welfare losses of a strong patent policy lie beyond the loss in consumer surplus due to monopoly power.

In this paper, I provide a tractable continuous-time model of R&D investments in an economy where innovation is sequential and has the Schumpeterian property of creative destruction. A patent is represented by a two-dimensional policy that determines how long an innovator will be able to exclude others from using his technology –i.e. patent length– and how much protection will an innovator have from future inventions –also called forward protection or patent breath. When a new invention occurs, the patent authority may determine that the new invention infringes the currently active patent. In that case, license fees, equal to the damages caused by the commercialization of the new product, have to be paid by the innovator in order to be able to commercialize his new invention and obtain economic profits.

Since patent protection expires, the value of an active patent decreases with the approximation of its expiration date. This induces R&D incentives –and consequently investments– to be non-stationary though time. Also, it makes patent policy play an important role in the *timing* of R&D investments. At any point in time, the prize obtained from an innovation corresponds to the *incremental rent* derived from a new patent. For an entrant, this rent corresponds to the value of a new patent minus the license fees he may have to pay in order to be able to commercialize his innovation. On the other hand, for the incumbent, the incremental rent corresponds to the difference between the value of a new patent and the cannibalized value of their currently active patent (this cannibalization is also known as Arrow's replacement effect, see Arrow (1962)).

The key insight gained from studying the problem in a sequential context is that patent policy not only affects the value of a new innovation, in equilibrium, it also affects the cost of replacing currently active patents. For instance, an increase in patent length will not only increase the value of developing a new patent –rising the firms' incremental rent, creating an incentive to invest more– it also increases the incumbent's valuation for his current patent and the expected license fees paid by innovating entrants. These effects, not present in a single-innovation context, discourage firms from innovating. Furthermore, when patent protection is too strong, these effects become dominant, decreasing the firms incremental rent, lowering investment rates and, ultimately, decreasing the economy's pace of innovation.

After understanding the firms' investment dynamics, I study the welfare implications of a patent system. To do so, I depart from the standard approach of assessing the consumers' dead weigh loss associated with the monopoly granted by patent protection and focus solely in the speed of the innovative activity induced by a given policy. In particular, I study how the optimal policy –understood as the one that maximizes the innovative activity– varies through different markets. If find that in markets in which innovations take on average longer to produce or are costly to generate, longer patents with little forward protection are optimal. In contrasts, market in which innovations occur frequently, short patents with a strong forward protection are optimal.

This paper represents a contribution to the innovation literature in three respects. First, I provide a clean theoretical model that sheds light on how the dynamic incentives for innovation evolve through the patent life. To my knowledge, this is the first paper that, in a context of sequential innovation, studies the incentives induced by *finite* patent lengths, looks at optimal patent policy, and performs comparative statics. Secondly, my model rationalizes some empirical findings, such as incumbents: invest less than entrants (Czarnitzki and Kraft 2004, Acs and Audretsch 1988), patent at lower rates (Bound *et al.*, 1984), and, further, invest at a slower pace (Igami, 2011). Moreover, it is consistent with the empirical evidence provided by Sakakibara and Branstetter (2001) that stronger patent policy does not necessarily lead to more R&D (see Cohen 2010 for exhaustive survey in the empirical R&D literature). Lastly, my findings bring new insights and provide deeper understanding of patent systems –the most common tool used by governments to promote innovation.

The paper is organized as follows: Section 2 presents a simple model of innovation which is the basis of the subsequent analysis. In Section 3, I prove the equilibrium's existence and find its analytical solution (Proposition 1). There, I show that the firms' investments increase with the approximation of the patent expiration date as both the incumbent's cannibalization of his own patent, and the license fees paid by innovating entrants, vanish with the decrease in value of the currently active patent.

In Section 4, as a comparative static analysis, I explore the firms investment dynamics. I start by studding how the different elements of the model determine the value of an active patent. Proposition 2 shows that higher expected discounted profits and stronger patent protection increases the value of active patents. Despite of this increase in value, stronger protection does not necessarily lead to higher investments. Proposition 4 studies the effect of patent length on firms R&D. Under longer patent protection, incumbents decrease their investment rates, postponing it towards the end of their patent protection. Entrants, on the other hand, purse the innovation harder at the beginning of incumbent's patent protection but may decline their efforts toward the end of the incumbent's protection. This decrease in investments is caused by the fact that, in a sequential context, patent policy not only affects the value of a new innovation but it also affects the replacement value of active patents, and, consequently potential license fees that entrants have to pay in case that an infringement occurs.

As Proposition 6 shows, incumbents and entrants also react asymmetrically to an increase in the level of forward protection that a patent grants. An increase in forward protection raises the incumbents incremental rent from innovating, leading incumbents to perform more R&D. In contrasts, due to an increase in the probability of infringing active patents, the increase in forward protection decreases entrants incremental rent at the beginning of the incumbent's protection, decreasing their R&D. Since the value of the license fees paid in case of an infringement vanish with the approximation of the patent expiration date, the deterrence effect from the increase in forward protection fades away towards the end of patent life, increasing the entrants R&D .

In Section 5 I perform numerical analysis to study the policy that maximizes the innovative speed of the economy. There I show that the optimal policy consist in a finite patent length and a non-maximal level of forward protection. This result contrast with the classical patent literature which, in the absence of consumers, infinite patent length would maximize the innovative activity in the economy. In this section, I also show that the optimal patent varies through different markets and that the two policy tools tend to substitute one another. In particular, in markets in which innovation occur frequently short patents with high levels of forward protection tend maximize the innovative activity. In contrasts, in slow innovating markets, long patents with little forward protection are optimal.

Section 6 presents different extensions that aim to understand the role of the different assumptions used in the basic model. Section 6.1 extends the model to the case in which license fees are determined through a bargaining process between the incumbent and the infringing entrant. There, I show that the larger the incumbent bargaining power the less effective becomes the patent's forward protection as a tool to promote innovation. In Section 6.2 I study the extent to which previous results are driven by the assumption that the commercialization of new inventions completely cannibalize profits from previous patents. As it should be expected, the less dynamic competition a patent face, the closer the model is to a single-innovation model, and the weaker the results become. Lastly, through numerical analysis, I examine a duopoly model to study the effects of strategic interaction between firms. As shown there, the conclusion of the basic model remain unaffected.

To conclude and due to the large extent of the previous literature, I devote Section 7 to discuss how my results relate to the previous work in patents and Section 8 concludes.

2 A Simple Model of Sequential Innovation

Consider an economy consisting of firms competing in an infinitely-lived market. Firms perform R&D in order to obtain an innovation that allow them to obtain temporary competitive advantages. Time is continuous and denoted by t. At each instant of time t, there is at most one firm possessing an active patent for the latest technology available in the market; this firm is called the *incumbent* and is denoted by i. All other firms, called *entrants* and denoted by n, compete investing in R&D in order to leapfrog the incumbent and become the new technology leader. I assume that the incumbent is a long-run player, who maximizes the present discounted value of his profits, whereas the entrants are an infinite sequence of short-run investors that maximize their instantaneous payoffs.[1] The incumbent discounts his future payoffs at a rate $r > 0$.

The value of an innovation depends on the underlying patent system, the monopoly profits that the incumbent gets while he holds the patent, and the time remaining until the patent's expiration date. A *patent system* consists of a *statutory length* $T \in \mathbb{R}^+$ and of *forward protection* represented by an infringement probability $b \in [0, 1]$. I assume that all innovations are patentable but a new innovation may infringe the previously active patent with probability b. While an incumbent's infringement of its own patent has no active consequences, entrants have to pay a compulsory license fee to be able to profit from their invention. The license fee is assumed to be equal to the profit damages that the incumbent suffers from the commercialization of the new product. For all the participants of this economy the tuple (T, b) is considered common knowledge and exogenously given.

While a patent is active, the incumbent receives a monopoly flow of profits $\pi > 0$. When the patent expires, competition in the product market drives the incumbent's profits to zero. When an innovation occurs, the inventing firm patents his technology right away, gaining the right to exclude others from using his technology at the cost of making this new technology publicly known. As a consequence, any innovation produced by an entrant will build upon the latest technology, leapfrogging the current incumbent. Furthermore, I assume that the new technology renders the currently patented technology obsolete, driving the profits of the replaced incumbent to zero. In concrete terms, the conjunction of these assumptions imply that there is, at most, a one-step lead between the technology leader and his competitors.[2]

A *period* is the time lapse between two innovations. Periods have random length and are basis of the notation below; $t = 0$ represents the beginning of a period and $t = s$ represents that s units of time have passed since the last innovation; that is, if $s < T$, the currently active patent has $T - s$ time left.

In order to obtain an innovation firms have to invest in R&D. These investments lead to a stochastic arrival of inventions which is an increasing function of the firms' investments. At every t, each firm $k \in \{i, n\}$ chooses an R&D effort $x_{k,t} \geq 0$. The instantaneous cost of this investment is given by the cost function $c(x)$. I assume that the cost is a twice continuously differentiable function that satisfies $c'(\cdot), c''(\cdot) > 0$, $c'(0) = 0$ and $\lim_{x\to\infty} c'(x) = \infty$ (in order to obtain an analytical solution below I will focus on the case in which $c(x) = x^2/2$.)

Firm's k effort induces an arrival of innovations described by a non-homogeneous Poisson process whose arrival rate at instant t is $\lambda x_{k,t}$ with $\lambda > 0$. The parameter λ is called the market's *natural innovation rate* and is a measure of the pace of innovation in any given market[3]. The Poisson processes are independent among firms and generate a stochastic process that is

[1] Alternatively, this assumption can be re-interpreted as free entry into the R&D race; see footnote 4.

[2] In section 6.2 I relax the assumption that a new innovation makes the current technology completely obsolete. As shown there, the main force driving the results is still present, but fades away with the decrease in rivalry among innovations.

[3] Comment about cost and λ

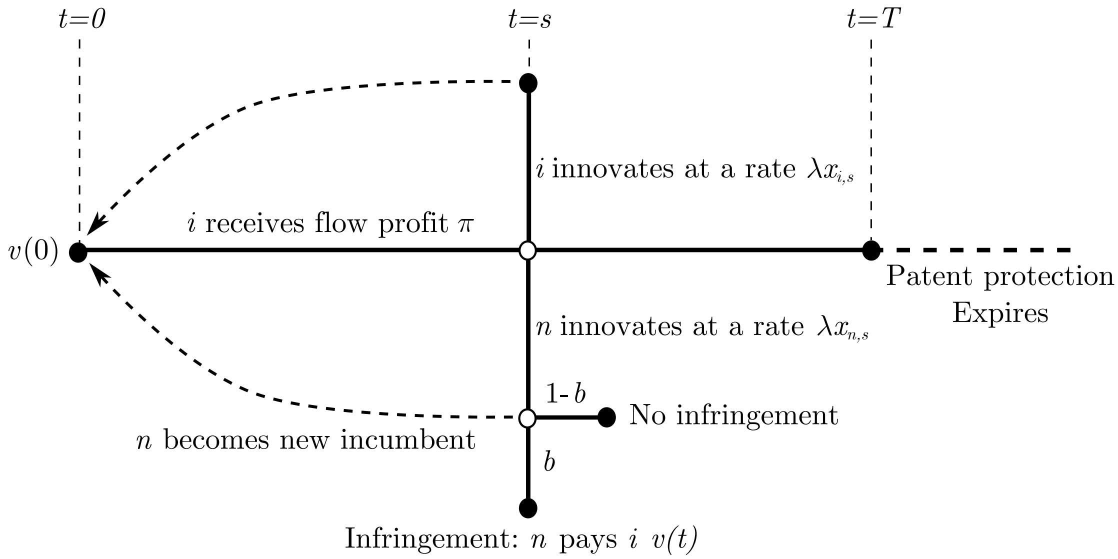

Figure 1: Timing of the game

memoryless but potentially non-stationary. The waiting time between two innovations is described by an exponential distribution with the probability of observing an innovation by instant t equal to $1 - \exp(-z_{0,t})$ where $z_{\tau,t} = \lambda \int_\tau^t (x_{i,s} + x_{n,s})\, ds$ is a measure of the accumulated investments from instant τ to instant t.

Timing of the game The timing of the game, depicted in Figure 2, is as follows. When an innovation arrives our time index t is reset to zero. From that time and on, and while his incumbency lasts, the patent holder receives a monopoly flow of π whereas the entrants obtain zero profit as they only have access to obsolete technologies. At each instant of time, the incumbent faces a different entrant. Entrants are assumed to play only once in the game maximizing their instantaneous payoff. Thus, the effort $x_{n,t}$ represents how the investment of the different entrants evolves through time.[4] At every instant of time, both incumbent and entrant choose their effort simultaneously determining the arrival rate of innovation for both firms.

Let $v(t)$ be the equilibrium value of holding a patent that has been active for $t < T$ years. When an innovation occurs, the succeeding firm becomes the new incumbent and his technology renders the currently patented technology obsolete. If the innovating player is an entrant, with exogenous probability b his innovation is considered to infringe the incumbent's patent and he has to pay a compulsory license fee of $v(t)$ equal to the damages caused to the incumbent due to

[4] This assumption can be thought of a reduced form of the following model: There are two arrival processes, one for the incumbent and one for the entrants. All the potential entrants play simultaneously. The arrival rate of the entrants depends on the sum of their efforts denoted by $x_{n,t}$. If the entrants' arrival process delivers an innovation, the succeeding firm is chosen with an uniform probability among all possible entrants. When the number of entrants is arbitrarily large, as it would be if there is free entry in the R&D race, each firm has efforts and payoffs that converge to zero but on the aggregate total effort, $x_{n,t}$, converges to that represented in equation (2) below.

the commercialization of the new invention. If no innovation has occurred within the statutory length of the patent, the patent holder loses his incumbency status and becomes one of the many entrants of the game. Consequently, no license fee can be charged for innovations that occur after T and the value of a patent at that date is zero, i.e. $v(T) = 0$.

Payoffs and strategies Fix $\{x_{n,t}\}_0^T$. From the perspective of time s, the incumbent's value of having a patent that has been active for s years is:

$$v(s) = \max_{\{x_{i,t}\}_s^T} \int_s^T (\pi + \lambda x_{i,t} v(0) + \lambda x_{n,t} b v(t) - c(x_{i,t})) e^{-z_{s,t}} e^{-r(t-s)} dt. \tag{1}$$

That is, with probability $\exp(-z_{s,t})$ no innovation has occurred between instant s and t and the patent is still active at t. At that instant t, the incumbent receives the flow payoff π and pays the cost of his investment $c(x_{i,t})$. The R&D investment leads him to an innovation at a rate $\lambda x_{i,t}$ obtaining the benefit of a brand new patent $v(0)$. On the other hand, the entrant may succeed at a rate $\lambda x_{n,t}$ in which case he may infringe the current patent with probability b, having to pay to the incumbent a compulsory license fee of $v(t)$. All of these payoffs are discounted by $e^{-r(t-s)}$.

On the other hand, at each instant of time t at which a patent is active, a new entrant decides how much to invest. Entrants maximize their instantaneous flow payoff

$$\lambda x_{n,t} ((1-b) v(0) + b(v(0) - v(t))) - c(x_{n,t}).$$

This is, the entrant's expected profit from an innovation $v(0) - bv(t)$, adjusted by the arrival rate induced by his effort $\lambda x_{n,t}$, minus the cost of his effort $c(x_{n,t})$. It is immediate to see that the entrant's effort at each instant of time will be given by

$$c'(x_{n,t}) = \lambda(v(0) - bv(t)). \tag{2}$$

Similarly, when no patent is active, and no license fee can be charged for an innovation, the entrant's effort becomes constant and equal to $x_{n,t} = c'^{-1}(\lambda v_0)$.

In this game a strategy is a mapping from the time that the current patent has been active, t, to a R&D intensity. Therefore, this is a Markov game as both the incumbent's and entrants' strategies are functions of the only state variable. Thus, I focus on finding Markov Perfect equilibria.

3 The Incumbent's Problem

In this section I solve the incumbent's problem using optimal control techniques. I start by assuming that the value of a new innovation is known, and equal to v_0.[5] Then, I apply the

[5]To be clear, $v(0)$ represents the value of an active patent that was just issued. On the other hand, v_0 represents the value of a new patent that has not been issued, i.e. is the value of the next innovation.

Principle of Optimality to derive the Hamilton-Jacobi-Bellman (HJB) equation which provides necessary and sufficient conditions for a maximum. Maximizing the HJB equation I find the incumbent's optimal investment rule which is used to solve for the value of having an active patent at each instant t. This solution is implicitly defined by the conjectured value v_0. Thus, this section concludes by showing that there exists a value of v_0 that is consistent with the solution found above, i.e. $v(0) = v_0$.

Let $x_t = x_{i,t} + x_{n,t}$, starting at an arbitrarily small time interval $[t, t + dt)$, the incumbent's value of having a patent for t years must satisfy the Principle of Optimality:

$$v(t) = \max_{x_{i,t}} \left\{ (\pi + \lambda x_{i,t} v_0 + \lambda x_{n,t} b v(t) - c(x_{i,t})) dt + e^{-rdt} \mathbb{E}\left[v(t + dt) \,|x_t \right] \right\}.$$

That is, given the optimal strategy, the value of having a patent at t is equal to the payoff flow at that instant of time, plus the discounted expected continuation value of the patent. For sufficiently small dt, the discount factor $\exp(-rdt)$ is equal to $1 - rdt$. On the other hand, the expected continuation value of the patent $\mathbb{E}\left[v(t + dt) \,|x_t \right]$ is equal to the probability of not having an innovation today $1 - \lambda x_t dt$, times the value of a patent tomorrow $v(t + dt) = v(t) + v'(t)\, dt$, plus the probability that an innovation occurs $\lambda x_t dt$ times the continuation value of the *current* patent after an innovation which is zero. Substituting back the previous expressions in the equation derived from the Principle of Optimality, I obtain the following HJB equation

$$rv(t) = \max_{x_{i,t}} \left\{ \pi + \lambda x_{i,t} (v_0 - v(t)) - \lambda x_{n,t} (1 - b) v(t) - c(x_{i,t}) + v'(t) \right\}. \tag{3}$$

Condition (3) is necessary and sufficient for a maximum, and the incumbent's optimal R&D intensity is determined by its first-order condition

$$c'(x_{i,t}) = \lambda (v_0 - v(t)). \tag{4}$$

Equations (2) and (4) are very informative about the firms' R&D investments dynamics. They state that, at any instant t, the firm's marginal benefit of their R&D investments is a function of the *incremental* rent that the firms obtain from innovating. For the incumbent this is the expected profits from a new patent v_0, minus the expected profits loss from giving up the currently active patent $v(t)$. For the entrant the incremental rent corresponds to the profits from a new patent minus the expected license fee $bv(t)$ that he has to pay in order to commercialize his innovation.

Since the value of an active patent declines with the proximity of its expiration date (see Proposition 1), both types of firms perform increasing investments through time. Also, before a patent expires and as long as $b < 1$, entrants invest at a higher rate than the incumbent. The incumbent starts performing zero effort at $t = 0$ as, by definition, $v(0) = v_0$, whereas the entrant investment starts at $x_{n,t} = c'^{-1}(\lambda(1 - b)v_0)$. Effort reaches its maximum at $t = T$, where the replacement value of the currently active patent becomes zero, and both firms invest at a rate of $c'^{-1}(\lambda v_0)$.

To be able to obtain an analytic solution more structure on the cost function is needed, in particular I assume the following quadratic cost $c(x) = x^2/2$. Substituting the incumbent's and entrant's effort into (3), using the cost assumption, and rearranging, the following ordinary differential equation (ODE) is obtained[6]

$$0 = v'(t) + \lambda^2 \left(\frac{1}{2} + b(1-b)\right) v(t)^2 - \left(r + \lambda^2 (2-b) v_0\right) v(t) + \pi + \frac{\lambda^2}{2} v_0^2. \tag{5}$$

This is a separable Riccati differential equation. It has a unique solution that satisfies the boundary condition that patents lose their value after they expire, $v(T) = 0$, and its solution is given by (see Appendix A for details)

$$v(t) = \frac{(2\pi + (\lambda v_0)^2)(e^{\phi(T-t)} - 1)}{(\theta + \phi) e^{\phi(T-t)} - (\theta - \phi)} \tag{6}$$

where $\theta = r + \lambda^2 (2-b) v_0$ and $\phi = \left(\theta^2 - \lambda^2 (1 + 2b(1-b)) \left(2\pi + \lambda^2 v_0^2\right)\right)^{1/2}$.

As shown by equation (6), the value of a patent $v(t)$ is a function of the conjectured value v_0. In order to have a well-defined solution it is necessary to show that a fixed point to $v(0) = v_0$ exists. It is important to note here that $v(t)$, as a function of v_0, is well-defined for all non-negative values of v_0. In particular, this is also true for values of the parameters such that ϕ is imaginary because all the imaginary terms in (6) cancel out (see Appendix A.) The next proposition establishes the existence of a solution.

Proposition 1 (Existence) *There always exists a fixed point $v(0) = v_0$. The value of having a patent decreases with t and is given by equation (6) evaluated at the fixed-point v_0. The incumbent's R&D investments increase as the patent expiration date approaches, whereas the entrant investments increase whenever $b > 0$. The R&D investments for the incumbent and the entrants are respectively given by*

$$x_{i,t} = \lambda(v_0 - v(t)) \qquad x_{n,t} = \lambda(v_0 - bv(t)).$$

The proof of Proposition 1 is relegated to Appendix B.1. Unfortunately, uniqueness of the fixed point cannot be guaranteed for all sets of parameters. However, since there is a finite number of fixed points and they can be ranked in a Pareto sense, i.e. all players rank the fixed points in terms of their payoffs equally, in the subsequent analysis I use the Pareto-dominant fixed point as the solution, i.e. the largest v_0 that is a fixed point.

4 R&D Dynamics

The purpose of this section is to study how the different elements of this economy affect the firms' R&D investments. Since investments are function of the value of a patent through time, a key

[6] This specification of equation (5) assumes that firm's investments are non-negative. This will be the case in equilibrium, however the conjectured value v_0 may be sufficiently low that the incumbent or both firms may choose no to perform R&D for some t. Appendix A provides details in this matter.

step to understand R&D dynamics is to study how the value of a patent $v(t)$ is affected by changes in the parameters or the model.

The value of an active patent In equilibrium, when innovation is sequential, exogenous changes in the parameters of the model will have two –often opposing– effects. On the one hand, there is a direct effect on the patent race taking place at the moment of the change. On the other, there is an indirect effect in the patent races taking place in the future. This last effect is captured by changes in the fixed-point $v(0) = v_0$, and generally makes the comparative static not as straight forward as ones would expect to be. In what follows, Proposition 2 summarizes the net equilibrium effect of a change in a parameter of the model, whereas Proposition 3 isolates the fixed-point effect.

Proposition 2 (Value of a new patent) *The equilibrium value of a patent $v(t)$ increases with:* (i) *an increase in the profit flow π;* (ii) *a decrease the interest rate r;* (iii) *an increase of the statutory length T;* (iv) *an increase of the forward protection b whenever $b < 1/2$, and;* (v) *an increase in the market's natural rate of innovation λ when there is full forward protection.*

Claims (i)-(iv) in Proposition 2 are quite intuitive. If the discounted flow of monopoly profits is higher or if patent protection is stronger, the equilibrium value of a patent goes up. Although the proof that greater forward protection increases the value of a patent holds only for $b < 1/2$, numerical results show that the increase in value is for all possible levels of forward protection (see Figure 2(a) for an example). This limitation in the proof is due to not having an explicit solution for $v(0)$ and a decrease in the expected revenues that an incumbent derives from the innovation of an entrant whenever $b > 1/2$. This revenue effect, studied in Section 6, pushes the value of an active patent down but seems to be dominated when license fees are equal to the damage suffered by incumbents due to the commercialization of a new innovation, but may dominate if we endow the incumbent with larger bargaining power.

The effect on an increase in the natural innovation rate λ is less clear as is the result of the interaction of two opposing forces. On the one hand, the increase in λ can be understood as an increase on the incumbent's competition as it increases the entrant's R&D productivity. On the other hand, the increase in λ also increases the incumbent's productivity of R&D, increasing his probability of remaining technology leader. Hence, the net effect of a change in λ depends on the relative magnitude of the increase-in-competition and the increase-in-productivity effects. Claim (v) states that when the incumbent possesses maximal forward protection, the increase on the incumbents productivity becomes the dominant effect, increasing the value of a patent. Although unproven, numerical analysis also show that the converse is also true. When no forward protection is offered, the competition effect becomes dominant and an increase in the natural innovation rate λ decreases the value of a new patent. As figure Figure 2(a) shows, for intermediate levels of forward protection the effect of an increase in λ may be non-monotonic and tend to decrease the

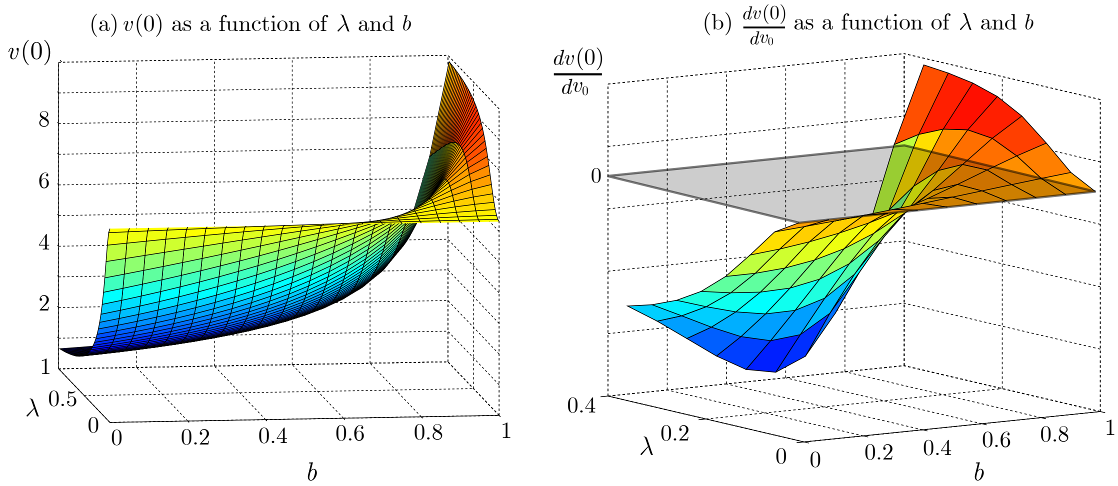

Figure 2: Comparative Statics: Value of a patent

value of a patent in markets in which innovations are frequent. In practical terms, this result implies that unless incumbents have strong protection against future inventions, they will not generally benefit with measures that facilitate innovation at an industry-wide level and this is particularly true in market in highly innovative markets.

Proposition 3 (Changes in future policy) *With no forward protection, an exogenous change in the value of the next technology, v_0, decreases the value of an active patent $v(t)$.*

Proposition 3 isolates the effect of anticipated changes in policy, i.e. changes that will will take place only after the next innovation occurs. As it happens with the natural innovation rate λ, a change in v_0 has an increase-in-competition effect and an increase-in-reward-from-R&D effect. Proposition 3 states that with no forward protection, the increase-in-competition effect dominates. Furthermore, with exception of full forward protection, numerical analysis show that increasing the value of future inventions decreases the value of currently active patents due to the increase in competition (see Figure 2(b)).

R&D Investments After understanding how the different aspects of the model affect the value of holding an active patent, I study how patent policy influence the dynamics of R&D investments.

Proposition 4 (Patent length and R&D) *An increase in the statutory length of a patent T:* (i) *decreases the incumbent's investment rate at each t, and;* (ii) *increases the entrants' investment at $t = 0$ but, when forward protection is large enough, may decrease its R&D for $t > 0$.*

At any instant of time, firms R&D investments are a function of the incremental rent they obtain from innovating (see equations (2) and (4)). For the incumbent this is the value of a new patent minus the cannibalized benefits of the old patent. For the entrant is the value of a new patent minus the license fees he has to pay in case of an infringement. As shown in Proposition

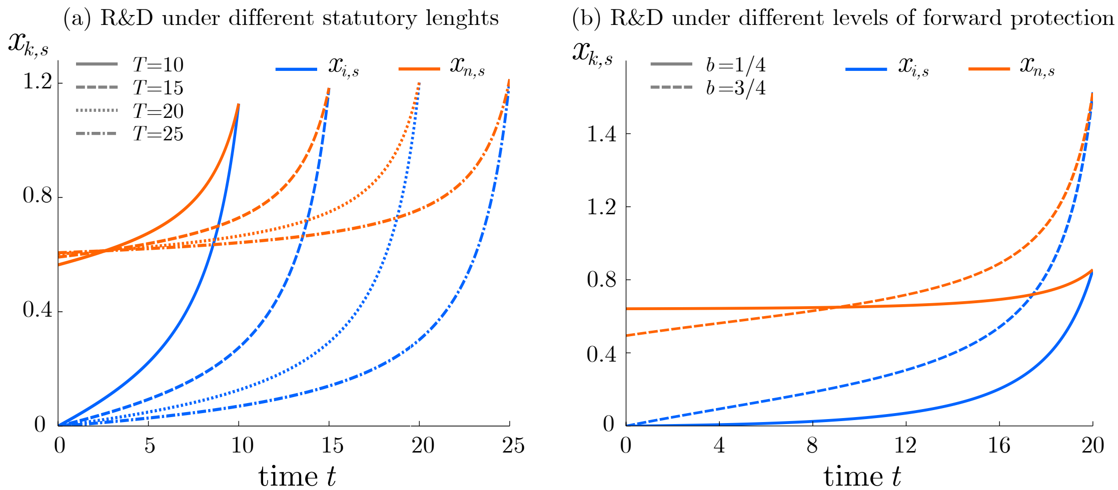

Figure 3: Comparative Statics: R&D investments and patent policy

2, an increase in the statutory length of a patent leads to an increase in the value of holding an active patent $v(t)$ for all t. Consequently, the equilibrium effect of an increase in patent length will depend on how the magnitude of the increase in $v(t)$ changes through the patent life t. Claim (i) in Proposition 4 implies that the increase in $v(t)$ is larger the closer the active patent is to its expiration date, this is due to the fact that the *effective* length of a patent generally differs from its *statutory* length. When longer patent protection is offered, the increase in value of an active patent at $t > 0$ is larger than the increase at $t = 0$ because the probability of actually reaching and using the patent extension is higher the closer the incumbent is to the expiration date T. Thus, the *effective* gain due to an increase in the duration of the patent is larger the closer the patent is to the initial expiration date.

To understand claim (ii) observe that the entrant's R&D can be written as

$$x_{n,t} = bx_{i,t} + \lambda(1 - b)v_0.$$

Thus, the total effect in the entrants' investments is a convex combination of the effect of patent length on the value of a new patent and on the effect on the incumbent's investments. When $t = 0$, $x_{i,0} = 0$ and the only effect present is the increase in the value of a new patent. As time goes through and since license fees correspond to the value of the cannibalized patent, the increase in the expected license fees derived from the increase in T becomes larger and, when b is large enough, dominates the increase in v_0 driving the entrants' investment down. Figure 3(a) depicts Proposition 4 when $b = 1/3$. As can be observed, the increase in license fee effect is quite strong and dominates even with low degrees of forward protection.

The next proposition connects the sequential model with traditional single-innovation models and helps to gain a deeper understanding of Proposition 4.

Proposition 5 (Grandfathering) *If an increase in the statutory length of a patent T do not*

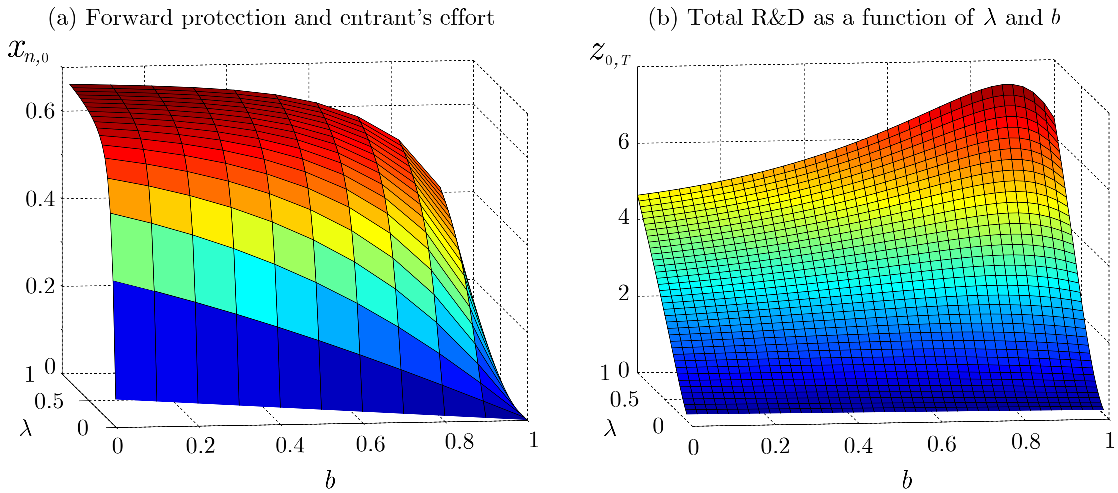

Figure 4: Comparative Statics: R&D investments and forward protection

apply to currently active patents but apply to all subsequent innovations, then incumbent and entrants will increase their R&D intensity at the moment of the change in policy.

Proposition 5 shows that Proposition 4 correspond to equilibrium effects, i.e. once all the effects in the change in policy have taken place. As single-innovation models do, when the analysis of an increase in patent length is restricted to the increase in the reward that firms obtain from future innovations v_0, we conclude that R&D should increase. Is precisely by incorporating the sequential structure to the standard innovation model that we find that the replacement value of an active patent, $v(t)$, also depends on patent policy, leading to the conclusion that R&D investments may decrease in equilibrium.

Proposition 6 (Forward protection and R&D) *An increase in forward protection b:* (i) *increases the incumbent's investment rate at each t, and;* (ii) *decreases the entrants' investment rate at $t = 0$ whenever $b \geq 1/2$, and increases the entrants' investments at the end of the patent life.*

In contrast to the case of a patent extension, the benefit of an increase in forward protection fades away when the patent expiration date approaches. In particular, at $t = T$ the effect of an increase in b is nil. As a consequence, the incumbent's incremental rent from innovating raises at each t, increasing his R&D investment rate. For entrants, on the other hand, there is a direct negative effect on their incremental rent due to the increase in the infringing probability. This effect leads to a decrease in the entrants' investment rates at the beginning of the patent protection. When the patent expiration date approaches, licenses fees decrease to zero. Hence, the increase in value of a new patent due to the increase in b dominates, increasing the entrants' R&D investment towards the end of patent life. These effects are shown in Figure 3(b), where I depict the firms investment dynamics for different levels of forward protection b.

To gain a better understanding on the role that forward protection plays in the firms investment dynamics, Figure 4 show some numerical results. Figure 4(a) suggests that the initial decrease in the entrants' investments due an increase in b is generally not restricted to $b > 1/2$. Also, the decrease of the initial investments is sharper the faster the speed of innovation in the economy is. Figure 4(b) shows the total accumulated investments throughout the patent life. Although subtle, for each λ, there generally is an interior level of forward protection that maximizes total investments. Moreover, this interior value of b is larger in economies that innovates faster.

The combination of Propositions 4 and 6 give clear and testable empirical predictions. First, the probability that an incumbent innovates in a technology that cannibalize market share from one of their current products increases with the approximation of the patent expiration date. Secondly, entrants are always more likely to innovate than incumbents, however, the the difference in probability converges to zero as the patent expiration date approaches. Third, the entrants' innovation rate is strictly related to the enforceability of a patent's forward protection. More precisely, we should observe more entrants innovating in markets were the degree of infringement of previous patents is harder to determine.

To conclude this section, I relate my model with the assumption more commonly used in the sequential innovation and grow literature.

Proposition 7 (Stationarity) *Under an infinite statutory patent length, the firms investments become constant. Incumbents perform no R&D whereas entrants' investment are* $x_n = \lambda(1-b)v_\infty$*, where* v_∞ *is the value of a patent, which is independent of* t *and equal to*

$$v_\infty = \frac{1}{2\lambda^2 (1-b)^2}\left(-r + \sqrt{r^2 + 4\pi\lambda^2 (1-b)^2}\right)$$

when $b < 1$*, and to* $v_\infty = \pi/r$ *when* $b = 1$*.*

When patent protection is infinitely long, incentives become stationary and incumbents have no incentives to innovate. This is due to the fact that a new innovation just replaces the currently active patent with one of the same value. The intuition for the stationary in the value of the patent is that, under infinite patent protection, the benefit from innovating does not change as time passes; at any point of the patent life the incumbent is protected for the same expected amount of time.

Proposition 7 also provides a better understanding, and in a sense justifies, the assumption of passive incumbents generally made in the growth and innovation literature.[7] Passive incumbents will only exist in environments in which patents have an infinite statutory length. Also, as shown in Section 6, the assumption that an innovation completely cannibalizes the value of an existent patent, is also necessary.

[7] Although previous proof of this result exists (see for example Aghion and Howitt (1992)), my framework provides a direct proof, helping to get a better understanding of the underlying mechanisms that drives it.

5 Optimal patent policy

In this section I study different policies in terms of their capacity to generate higher innovation rates. In particular, I study the policy that maximizes the innovative activity and how it varies across markets according to their intrinsic capacity to produce innovations. The purpose of following this methodology, and not maximizing total welfare, is to understand the incentives that patent policy imposes on firms and the role that the different elements in patent policy play at incentivizing firms. Another motivation for this methodology is that much of the applied work and policy discussions focus on the speed of the innovative activity rather than total welfare. This is so as, in principle, the former is easier to measure than the latter.

To define our measure of innovative activity I leverage from the property that innovations follow a non-homogeneous exponential distribution. In particular, I study the policy (T^*, b^*) that maximizes economy's expected arrival time of innovations $\hat{\lambda}$ which is defined as[8,9]

$$\hat{\lambda} = \mathbb{E}[t]^{-1} = \left(\int_0^\infty \lambda x_t t e^{-z_{0,t}} dt \right)^{-1}.$$

Result 8 (Optimal patent policy) *The expected innovation rate of the economy, $\hat{\lambda}$, is maximized at a finite patent length and at a non maximal forward protection level. The optimal length, T^*, decreases with the natural innovation rate of the economy λ, whereas the optimal forward protection, b^*, increases with it.*[10]

Figure 5(a) exemplifies how the expected arrival rate of the economy varies with different policies. As can be observed, $\hat{\lambda}$ is maximized at an interior point of the policy space. Fixing a level of forward protection b we learn that the effect of patent length on the speed of innovation is non-monotonic. In Proposition 2 we learned that the value of a patent is increasing in the length of patent protection; hence, with low values of T, rewards are too low to induce high rates of innovation (for instance, in the limit, when $T = 0$ no R&D is performed.) In contrast, when patent protection last too long, Proposition 4 teaches us that incumbent delay their investments which, at some point, will induce the expected innovation speed to decrease.

[8]For the purpose of illustration, if the total investment x_t where constant through time and equal to one, the distribution of successes will follow an exponential distribution with an arrival rate equal to λ. In that case we have $\hat{\lambda} = (\mathbb{E}[t])^{-1} = \lambda$, and the expected innovation rate of the economy corresponds exactly to its natural innovation rate.

[9]In many games in which the reward from effort is random (see Reinganum (1982) or more recently Keller *et al.* (2005)) the innovation rate is approximated by the total investment performed during the game. This methodology, however, is not an appropriate measure for a game of infinite length as investments never stop. Another option is to restrict attention to the total investments performed during patent protection $z_{0,T}$. However, this methodology ignores the R&D performed after the expiration of a patent which is also affected by patent policy. Other authors like Bonatti and Hörner (2011) have used stochastic dominance (SD) to compare the distribution of successes for different sets of parameters. Here, the lack of monotonicity in the comparison of policies makes SD uninformative.

[10]The term Result is used to make the distinction that this is a numerical result without proof.

(a) Optimal policy has an interior solution

(b) Optimal patent policy as a function of λ

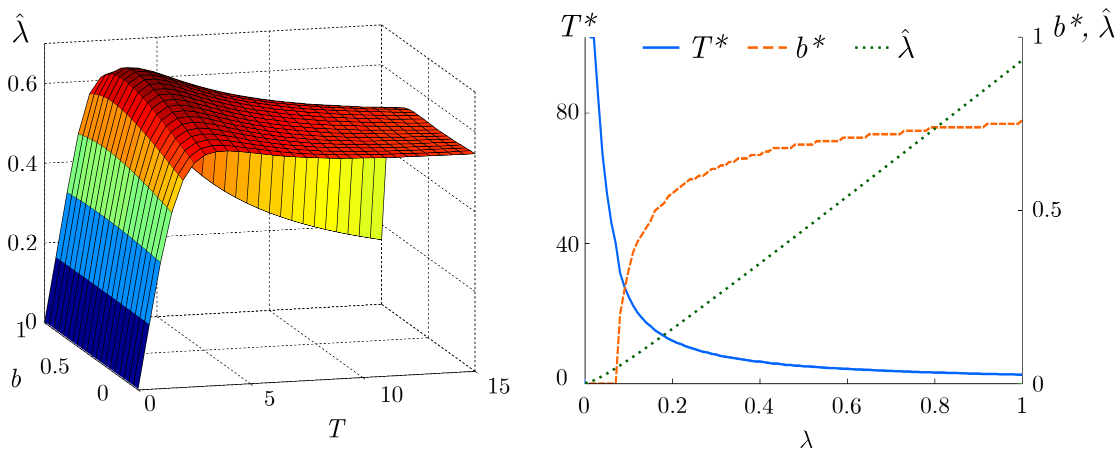

Figure 5: Comparative Statics: R&D investments and forward protection

The effect of forward protection in the innovative speed is related to the length of the patent. Longer patent protection delays the incumbent investment, and the investments performed by entrants at the beginning of the active patent life become important to determine the speed on the innovative activity. By Proposition 6, higher levels of forward protection decreases entrants' investments at the beginning of the patent race. Thus, as shown in Figure 5(a), higher levels of forward protection will have a larger negative impact in the speed of innovation when patent protection last for too long.

Figure 5(b) show how the policy that maximizes the economy's innovation rate varies with the natural rate of innovation. The optimal patent length decreases with the market's natural innovation rate, whereas the maximizing forward protection increases. To understand the intuition of this result we have to compare the incentives present in markets that innovate fast, with those incentives present in market that innovate more lowly. When the natural innovation rate is high, innovation occurs at a higher frequency shortening the patent's effective life. In that case, patent length is an ineffective tool to encourage innovation as the reward from a new innovation, v_0, is inelastic with respect to changes in T. In this context and by making harder to entrants to obtain rents from innovating, forward protection acts as an effective way to extend the *effective* duration of a patent, increasing the rewards from innovation and incentivizing firms, in the aggregate, to perform higher R&D levels. In contrast, in markets in which innovation occur lest frequently, the value of a project v_0 is quite elastic to the statutory length of the patent, becoming a good tool to promote R&D. However, since longer patents leads incumbents to delay their investments, the entrant's innovation is crucial to speed the innovative activity up. Thus, little forward protection has to be offered in order to have firms performing R&D in early stages of the patent life and increase the speed of innovation.

At this point is important to contrast these results with those in the previous literature. In the single-innovation literature the only effect of increasing patent protection is to increase the reward from innovating, increasing R&D from both incumbents and entrants. Thus, in that context, the patent policy that maximizes the speed of innovation would contain infinitely long patents. Hence, the only reason to have finite patent is to limit the inefficiency created by the monopoly that the patent creates. In the context of a sequential model, long patents decreases the speed of the innovative activity creating a new source of inefficiency in addition to that derived from consumers.

Although forward protection has been studied in the sequential innovation literature, one the insights that we gain from incorporating the length of the patent in to the analysis, is to learn the interaction between the policy tools at the moment of given incentives. In particular, how the two policy tools substitute one another, and the way that optimal patents varies through markets seems to be a novel insight which contributes in the understanding of the dynamic incentives provided by patent policy.

6 Extensions

In this section I study new issues that may raise in the context of sequential innovation and the robustness of the results to some of the assumption in the model.

6.1 License Fees: The Hold-up Problem

6.2 Profit Cannibalization

6.3 Strategic Entrants

7 Relation with Previous Literature

The purpose of this section is to relate previous findings to the literature on patent policy and innovation. For clarity, the discussion is divided into a comparison of my results to those in the single-innovation literature, and to those in the sequential innovation literature. Finally, I discuss how this work relates to the literature of innovation and growth.

Single-innovation models and patent policy In the seminal work of Nordhaus (1969), inventions are produced by a deterministic production function under which innovations of higher quality have higher production costs. The main conclusion of Nordhaus's work is that longer patent protection induces inventions of higher quality but a greater social cost of monopoly. The interaction of these two effects calls for finite patent protection.

In contrast, my model belongs to a class of stochastic innovation. In these models inventions of a predetermined quality arrive stochastically and as a function of the firms' investment. Similar in spirit to Nordhaus, in the context of a single-innovation, Loury (1979) and Lee and Wilde (1980) find that longer patents increase the speed of innovation. Also, Denicolò (1999) makes the point that optimal patent length is finite due to the cost in consumer surplus. With respect to the single-innovation literature the key insight gained from analyzing the effects of patent length in a sequential world is that longer patents do not necessarily lead to higher innovation rates and therefore the cost associated with stronger patent policy may well be beyond the consumer surplus losses emphasized by the literature described above.

Another difference generated by the sequential framework is that incentives become non-stationary through time. In particular, investments increase as the patent expiration date approaches. In most of the single-innovation literature, patent races are stationary, meaning that a constant effort rate is performed by all the firms until an innovation occurs (an exception is Reinganum (1982) where the non-stationarity comes from the assumption that innovations may be only generated within a predetermined period of time). Even though the non-stationary property increases the difficulty of the analysis, it has the advantage of generating more realistic effort dynamics.

Earlier models of innovation, mainly inspired by Schumpeter (1942), studied the difference in incentives that incumbents and potential entrants may face (see Gilbert (2006) for a comprehensive theoretical and empirical review of the subject). Arrow (1962) showed that, when innovation cannibalizes part of an incumbent's profits, incumbents have less incentives to innovate than entrants. This effect is typically called the replacement effect. In contrast, Gilbert and Newbery (1982) argued that in an auction-like environment, where the firm that makes the highest investment/bid is the one that gets the patent, incumbents may be willing to outbid entrants due to the fact that the former may have more to lose than the latter to gain. One of my main contributions is to show that Arrow's effect is the key determinant of the incumbent's investment dynamics, and that patent policy not only affects the value of a new patent, it also affects the value of the replacement effect predicted by Arrow. Furthermore, is precisely the role of patent policy in determining the value of the replacement effect, which causes the decrease the economy's innovation rate for long patent protection.

Another branch of the single-innovation literature assumes that the costs and benefits of an innovation are known. In that respect, optimal patent policy consists of maximizing social welfare subject to an innovator-breaks-even constraint (Gilbert and Shapiro (1990); Klemperer (1990) and, in an environment with imitation, Gallini (1992)). This type of analysis is precluded in a sequential-stochastic world as both the cost of an innovation and the benefit of incumbency are random.

Policy in Sequential Innovation In spirit, the closest work to mine is Bessen and Maskin (2009). In a model of sequential innovation, where firms decide whether or not to undertake an innovative activity, they compare the situation in which either an infinitely long patent protection exists, or it does not. They find that patent protection may slow down the innovation process. The main contrast of their model to mine is that in my work I allow for patents of any length, not only the polar cases that deliver stationary investments. Also, in my framework, patents are necessary to promote innovation as the absence of patent protection drives the profits from an innovation to zero. In contrast, in their model, firms get positive profits regardless of the protection provided. Finally, while in their framework innovation does not cannibalize profits derived from previous innovations, my model is one of Creative Destruction where the cannibalization is total. In the substantive side, their result is driven by the existence of asymmetric information in the opponent's cost of R&D. In contrast, my result is driven by the role that patent policy plays in determining the incumbent's opportunity cost of innovation. Thus, both set of results complement each other by indicating different channels through which long patents may harm the innovation process.

Most sequential models focus on modeling different aspects of patent breadth (see Scotchmer (1991) for an insightful narrative paper that explains the general trade-offs explored in this literature). The early work of Scotchmer and Green (1990) and Scotchmer (1996) studies how patentability of second-generation products affects the development of first-generation products. Green and Scotchmer (1995) study the role of ex-ante and ex-post licensing agreements on R&D incentives.

A more recent literature initiated by O'Donoghue, Scotchmer and Thisse (1998) studies the effect of patents' novelty requirements on the market innovation rate in the context of an infinitely-lived market. These papers generally assume an infinitely-long patent protection and use patent breadth as a mechanism to deter inventions of lower quality. The lack of novelty requirements leads to fast imitation and to a reduction of the expected benefits of an innovation. Therefore, it is found that some breadth encourages innovation. Optimally, as shown by Hopenhayn *et al.* (2006), the minimal improvement required to obtain a patent is an increasing function of the quality of the previous innovation. I complement the study of patent systems by incorporating the effect of finite statutory lengths into the analysis.

My main contribution to the patent breadth literature is to highlight that its effectiveness depends on the patent statutory length and on the characteristics of the economic environment. In general, patent length is a more effective tool when the natural innovation rate of the market is low. Breadth becomes the predominant tool to encourage innovation when the gap between the the effective patent length and the statutory length renders patent length ineffective. This gap is a function of the market's natural innovation rate. In particular, when innovations are frequent by nature, patent length loses it effectiveness as innovations consistently arrive before the patents' statutory length. In that circumstance, breadth becomes the predominant tool as helps the planner to extend the effective length of a patent by making harder to overcome the incumbent's innovation.

More recently, there has been effort in understanding how other elements of policy affect innovation. For instance, Segal and Whinston (2007) study how antitrust affects the rate of innovation in a given market. My work complements this literature by deepening the understanding of how the different policy tools available can affect innovation.

Innovation, Market Structure and Growth There is a large literature that studies how different elements of the economy affect economic growth in a context where growth is mainly driven by the arrival of new innovations (see for example Grossman and Helpman (1991), and Aghion and Howitt (1992).) Although the focus of this work its far from that literature. I believe that this may be a first step towards modeling how patent policy may affects economic growth and competition between different countries. In particular, my research could be directly applied to a context in which innovations go through a quality ladder under the restriction of looking at one particular market instead of the economy as a whole.

8 Conclusion

This work developed a tractable model of sequential innovation, and studied how patent policy affects the R&D investment dynamics in an economy described by a quality ladder and by the Shumpeterian property of creative destruction. Due to profit cannibalization, I found that incumbents delay their investment towards the end of their patent protection as the benefit of an innovation increases with the approximation of the patent expiration date. This result is consistent and provides an explanation to the empirical evidence that incumbents: invest less in R&D, patent less, and adopt new technologies later.

I found that patent policy plays an important role in determining the rents of an innovation. In particular, patent policy not only determines the value of obtaining a new patent, it also determines the value of replacing the active patent held by the incumbent. For instance, an increase in patent length increases the value of an active patent more than the value of a new patent, decreasing the incumbent's prize from an innovation. As a consequence, at each point in time, the incumbent's effort rate decreases when offered longer protection, inducing the economy's innovation rate to react non-monotonically to changes in patent length. In particular, a patent extension increases the economy's innovation rate when the initial statutory length of a patent is short, and the extension decreases the economy's innovation rate when the initial statutory length is long. Therefore, the common result that stronger patent protection encourages growth is shown not to apply to this dynamic setting.

Important questions about how patent policy affects innovation in a sequential context remain open. For instance, firms do not always disclose their inventions right after developing them. For incumbent's, this may be done to avoid cannibalization of their profit flow. Entrants, on the other hand, may do it to avoid potential license fees from soon-to-be-expired patents. These potential delays of disclosure come at a cost; if the competition were to release the new technology first, the firm that waited may lose its potential benefit of being the next incumbent. Patent systems play an important role in the disclosure of innovations. If two firms where to claim similar technology to the patent office, in a first-to-invent system, the office will rule in favor of the innovator that invented a patent first but chose to not disclose his invention. In contrast, in a first-innovator-to-file system, the patent office will tend to rule in favor of the first firm that arrived at the patent office. I believe that my model could provide a basis to begin answering this or other questions about sequential innovation.

References

Acemoglu, D., Bimpikis, K. and Ozdaglar, A. (2011). Experimentation, patents, and innovation. *American Economic Journal: Microeconomics*, **3** (1), 37–77.

Acs, Z. J. and Audretsch, D. B. (1988). Innovation in large and small firms: An empirical analysis. *American Economic Review*, **78** (4), 678–90.

Aghion, P., Bloom, N., Blundell, R., Griffith, R. and Howitt, P. (2005). Competition and innovation: An inverted-u relationship. *The Quarterly Journal of Economics*, **120** (2), pp. 701–728.

—, Harris, C., Howitt, P. and Vickers, J. (2001). Competition, imitation and growth with step-by-step innovation. *The Review of Economic Studies*, **68** (3), pp. 467–492.

— and Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, **60** (2), pp. 323–351.

Arrow, K. (1962). Economic welfare and the allocation of resources for invention. In *The Rate and Direction of Inventive Activity: Economic and Social Factors*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 609–626.

Bessen, J. and Hunt, R. M. (2007). An empirical look at software patents. *Journal of Economics & Management Strategy*, **16** (1), 157–189.

— and Maskin, E. (2009). Sequential innovation, patents, and imitation. *The RAND Journal of Economics*, **40** (4), pp. 611–635.

Bonatti, A. and Hörner, J. (2011). Collaborating. *American Economic Review*, **101** (2), 632–63.

Bound, J., Cummins, C., Griliches, Z., Hall, B. and Jaffe, A. (1984). Who does r&d and who patents? In *R & D, Patents, and Productivity*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 21–54.

Chang, H. F. (1995). Patent scope, antitrust policy, and cumulative innovation. *The RAND Journal of Economics*, **26** (1), pp. 34–57.

Choi, J. P. (1991). Dynamic r&d competition under "hazard rate" uncertainty. *The RAND Journal of Economics*, **22** (4), pp. 596–610.

Cohen, W. M. (2010). Chapter 4 - fifty years of empirical studies of innovative activity and performance. In B. H. Hall and N. Rosenberg (eds.), *Handbook of The Economics of Innovation, Vol. 1*, *Handbook of the Economics of Innovation*, vol. 1, North-Holland, pp. 129 – 213.

Czarnitzki, D. and Kraft, K. (2004). An empirical test of the asymmetric models on innovative activity: who invests more into r&d, the incumbent or the challenger? *Journal of Economic Behavior & Organization*, **54** (2), 153–173.

D'Aspremont, C. and Jacquemin, A. (1988). Cooperative and noncooperative r & d in duopoly with spillovers. *The American Economic Review*, **78** (5), pp. 1133–1137.

Denicolò, V. (1996). Patent races and optimal patent breadth and length. *The Journal of Industrial Economics*, **44** (3), pp. 249–265.

Denicolò, V. (1999). The optimal life of a patent when the timing of innovation is stochastic. *International Journal of Industrial Organization*, **17** (6), 827–846.

Denicolò, V. (2000). Two-stage patent races and patent policy. *The RAND Journal of Economics*, **31** (3), pp. 488–501.

Denicolò, V. and Zanchettin, P. (2002). How should forward patent protection be provided? *International Journal of Industrial Organization*, **20** (6), 801–827.

Doraszelski, U. (2003). An r&d race with knowledge accumulation. *The RAND Journal of Economics*, **34** (1), pp. 20–42.

Farrell, J. and Shapiro, C. (2008). How strong are weak patents? *The American Economic Review*, **98** (4), pp. 1347–1369.

Gallini, N. T. (1992). Patent policy and costly imitation. *The RAND Journal of Economics*, **23** (1), pp. 52–63.

Gilbert, R. (2006). Looking for mr. schumpeter: Where are we in the competition-innovation debate? In *Innovation Policy and the Economy, Volume 6*, NBER Chapters, National Bureau of Economic Research, Inc, pp. 159–215.

— and Newbery, D. M. G. (1982). Preemptive patenting and the persistence of monopoly. *The American Economic Review*, **72** (3), pp. 514–526.

— and Shapiro, C. (1990). Optimal patent length and breadth. *The RAND Journal of Economics*, **21** (1), pp. 106–112.

Green, J. R. and Scotchmer, S. (1995). On the division of profit in sequential innovation. *The RAND Journal of Economics*, **26** (1), pp. 20–33.

Grossman, G. M. and Helpman, E. (1991). Quality ladders in the theory of growth. *Review of Economic Studies*, **58** (1), 43–61.

Harris, C. and Vickers, J. (1987). Racing with uncertainty. *The Review of Economic Studies*, **54** (1), pp. 1–21.

Hopenhayn, H., Llobet, G. and Mitchell, M. (2006). Rewarding sequential innovators: Prizes, patents, and buyouts. *Journal of Political Economy*, **114** (6), 1041–1068.

Hunt, R. M. (2004). Patentability, industry structure, and innovation. *The Journal of Industrial Economics*, **52** (3), pp. 401–425.

Igami, M. (2011). Estimating the innovators dilemma: Structural analysis of creative destruction in the hard disk drive industry, manuscript, Yale University.

Katz, M. L. and Shapiro, C. (1985). On the licensing of innovations. *The RAND Journal of Economics*, **16** (4), pp. 504–520.

— and — (1987). R and d rivalry with licensing or imitation. *The American Economic Review*, **77** (3), pp. 402–420.

Keller, G., Rady, S. and Cripps, M. (2005). Strategic experimentation with exponential bandits. *Econometrica*, **73** (1), 39–68.

Klemperer, P. (1990). How broad should the scope of patent protection be? *The RAND Journal of Economics*, **21** (1), pp. 113–130.

Lee, T. and Wilde, L. L. (1980). Market structure and innovation: A reformulation. *The Quarterly Journal of Economics*, **94** (2), pp. 429–436.

Loury, G. C. (1979). Market structure and innovation. *The Quarterly Journal of Economics*, **93** (3), pp. 395–410.

Malueg, D. A. and Tsutsui, S. O. (1997). Dynamic r&d competition with learning. *The RAND Journal of Economics*, **28** (4), pp. 751–772.

Mukherjee, A. and Pennings, E. (2004). Imitation, patent protection, and welfare. *Oxford Economic Papers*, **56** (4), pp. 715–733.

Nordhaus, W. (1969). *Invention growth, and welfare: a theoretical treatment of technological change*. M.I.T. monographs in economics, M.I.T. Press.

O'Donoghue, T. (1998). A patentability requirement for sequential innovation. *The RAND Journal of Economics*, **29** (4), pp. 654–679.

—, Scotchmer, S. and Thisse, J.-F. (1998). Patent breadth, patent life, and the pace of technological progress. *Journal of Economics & Management Strategy*, **7** (1), 1–32.

Reinganum, J. F. (1982). A dynamic game of r and d: Patent protection and competitive behavior. *Econometrica*, **50** (3), pp. 671–688.

— (1983). Uncertain innovation and the persistence of monopoly. *The American Economic Review*, **73** (4), pp. 741–748.

— (1985). Innovation and industry evolution. *The Quarterly Journal of Economics*, **100** (1), pp. 81–99.

Sakakibara, M. and Branstetter, L. (2001). Do stronger patents induce more innovation? evidence from the 1988 japanese patent law reforms. *The RAND Journal of Economics*, **32** (1), pp. 77–100.

Salant, S. W. (1984). Preemptive patenting and the persistence of monopoly: Comment. *The American Economic Review*, **74** (1), pp. 247–250.

Schumpeter, J. A. (1942). *Capitalism, socialism and democracy / by Joseph A. Schumpeter*. George Allen & Unwin, London :, 4th edn.

Scotchmer, S. (1991). Standing on the shoulders of giants: Cumulative research and the patent law. *The Journal of Economic Perspectives*, **5** (1), pp. 29–41.

— (1996). Protecting early innovators: Should second-generation products be patentable? *RAND Journal of Economics*, **27** (2), 322–331.

— and Green, J. (1990). Novelty and disclosure in patent law. *The RAND Journal of Economics*, **21** (1), pp. 131–146.

Segal, I. and Whinston, M. D. (2007). Antitrust in innovative industries. *American Economic Review*, **97** (5), 1703–1730.

Segerstrom, P. S. (1991). Innovation, imitation, and economic growth. *Journal of Political Economy*, **99** (4), pp. 807–827.

A Solution to ODE

In this appendix I solve the differential equation that describes how the value of a patent evolves with the proximity of its expiration date. Depending on the conjectured value v_0, competition during the life of the patent my go through three phases. Phase 0 happens when the value v_0 is low, so that $v_0 < bv(t) \leq v(t)$ and no firm will invest in R&D as the cost of replacing the currently active patent is larger than its benefit. Phase 2 occurs when $bv(t) \leq v_0 \leq v(t)$, i.e. when only the entrant has incentives to performs R&D. Finally, phase 2 occurs when $v_0 \geq v(t)$. In this case both will firms invest. In equilibrium, only phase 2 will be observed. However, for the purpose of proving existence, the three phases have to be characterized. Let $v_j(t)$ be the value of having an active patent in phase $j \in \{\emptyset, 1, 2\}$.[11] Henceforth, the expression $v(t)$ denotes $v_j(t)$ where j is the phase of the patent race at instant t.

Restate the differential equation (5), corresponding to phase 2, as

$$\frac{dv_2(t)}{dt} + av_2(t)^2 - \theta v_2(t) + \hat{a} = 0$$

where

$$a = \lambda^2(\frac{1}{2} + b(1-b)), \ \theta = r + \lambda^2(2-b)v_0, \text{ and } \hat{a} = \pi + \frac{\lambda^2}{2}v_0^2.$$

This ODE is separable and of the form $dv/h\,(v) = dt$ where $h\,(v) = -(av^2 - \theta v + \hat{a})$. Separable ODEs have unique non-singular solution that goes through its boundary condition, in this case $v_2(T) = 0$.[12] To find the non-singular solution I integrate both sides to get

$$-\ln\left(\frac{\theta - 2v_2(t)a + \sqrt{\theta^2 - 4a\hat{a}}}{\theta - 2v_2(t)a - \sqrt{\theta^2 - 4a\hat{a}}}\right)\sqrt{\frac{1}{\theta^2 - 4a\hat{a}}} = \hat{C} + t$$

where $\hat{C}$ is a constant of integration. Define $\phi = (\theta^2 - 4a\hat{a})^{1/2}$ and solving for $v(t)$ we find

$$v_2(t) = \frac{1}{2a}\left(\theta + \phi\frac{\left(1 + e^{-\phi(\hat{C}+t)}\right)}{\left(1 - e^{-\phi(\hat{C}+t)}\right)}\right) \tag{7}$$

which is the general solution to the ODE. To find the particular solution we just make use of the boundary condition $v_2(T) = 0$ to get

$$\hat{C} = -\frac{1}{\phi}\ln\left(\frac{\theta + \phi}{\theta - \phi}\right) - T. \tag{8}$$

Replacing back (8) in to (7) and rearranging terms we obtain

$$v_2(t) = \frac{\left(2\pi + \lambda^2 v_0^2\right)\left(e^{\phi(T-t)} - 1\right)}{(\theta + \phi)e^{\phi(T-t)} + \phi - \theta}$$

[11] I use $\emptyset$ to denote phase 0, so $v_\emptyset(t)$ does not get confused with the conjectured value v_0.

[12] Singular solutions to (5) are found by setting $dv/dt = 0$ and solving the quadratic equation. These solutions are disregarded as they do not generically satisfy the boundary condition $v\,(T) = 0$ and have no economic meaning.

which corresponds to equation (6). Now, I make sure that $v_2(t)$ is well defined for all positive conjectures of v_0. This clearly is true in cases where v_0 is such that $\phi > 0$. I have to check the cases under which ϕ is either imaginary or zero. For the former case, let $\phi = qi$ where q is the positive real coefficient of i and rewrite $v_2(t)$ as

$$v_2(t) = \frac{2\pi + \lambda^2 v_0^2}{\theta + q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i}.$$

Observe that Euler's identity implies[13]

$$q\frac{e^{q(T-t)i}+1}{e^{q(T-t)i}-1}i = \frac{q\sin\left(q\left(T-t\right)\right)}{1-\cos\left(q\left(T-t\right)\right)} \tag{9}$$

establishing that the value of a patent $v_2(t)$ is real when ϕ is imaginary.

Finally, for the case when $\phi = 0$, let $\hat{v}$ be the value of v_0 such that $\phi\left(\hat{v}\right) = 0$. When $\phi = 0$ the value of the patent at every t becomes $v_2(t) = 0/0$. Then, I define $v_2(t)$ to be the $\lim_{v_0 \to \hat{v}} v_2(t)$ which can be computed by applying L'Hôspital's rule to equation (6) and is equal to [14]

$$v_2\left(t\right) = \frac{\left(2\pi + \lambda^2 v_0^2\right)\left(T-t\right)}{\theta(T-t)+2}$$

showing that $v_2(t)$ is well defined for any possible value of v_0.

Similar steps can be follow to obtain $v_1(t)$; however, two key differences apply. First, the optimal investment rate of the incumbent is zero. Secondly, there exists $t_2 \leq T$ that determines the time in which phase 1 finishes and phase 2 starts, at that point the boundary condition $v_1(t_2) = v_2(t_2)$ must hold. Under those conditions I find

$$v_1(t) = \frac{v_2\left(t_2\right)\left(\theta_1 + \phi_1 + \left(\phi_1 - \theta_1\right)e^{\phi_1(t_2-t)}\right) + 2\pi\left(e^{\phi_1(t_2-t)} - 1\right)}{\phi_1\left(1 + e^{\phi_1(t_2-t)}\right) + \left(\theta_1 - 2a_1 v_2\left(t_2\right)\right)\left(e^{\phi_1(t_2-t)} - 1\right)}$$

where $a_1 = \lambda^2 b(1-b)$, $\theta_1 = r + \lambda^2(1-b)v_0$ and $\phi_1 = (\theta_1^2 - 4a_1\pi)^{1/2}$. Finally, the value of $v_\emptyset(t)$ is

$$v_\emptyset(t) = \frac{\pi}{r}\left(1 - e^{-r(t_1-t)}\right) + v_1(t_1)e^{-r(t_1-t)}$$

where $t_1 \leq T$ is the instant of time in which phase 1 starts. To conclude, t_1 and t_2 are found by solving $bv_1(t_1) = v_0$ and $v_2(t_2) = v_0$.

B Omitted Proofs

B.1 Proof of Proposition 1

I start by proving existence. From appendix A we know that there is a unique solution to the ODE, so I can restrict attention to show that exists a fixed-point $v(0) = v_0$ for a positive value of

[13]Euler's identity: $e^{i\psi} = \cos(\psi) + i\sin(\psi)$.

[14]In this case, left and right limit converge to the same point, so this is a well defined construction.

v_0.[15] To do so, I start by reformulating the problem defining a function $f(z) = v(0, z) - z$ where $v(0, z)$ denotes the dependence of the solution (6) on the conjectured value z. Then, showing the existence of the fixed-point is equivalent to show $f(z) = 0$.

I show existence by means of the intermediate value theorem. Observe that ϕ goes to ∞ when z goes to infinity. Then, it is easy to check that

$$\begin{aligned}\lim_{z\to\infty} f(z) &= \lim_{z\to\infty} \frac{\left(2\pi - z^2\lambda^2(1-b) - rz\right)\left(1 - \frac{1}{e^{\phi T}}\right) - z\phi\left(1 + \frac{1}{e^{\phi T}}\right)}{\theta\left(1 + \frac{1}{e^{\phi T}}\right) + \phi\left(1 + \frac{1}{e^{\phi T}}\right)} \\ &= -\infty.\end{aligned}$$

It remains to show that there exists z such that $f(z) > 0$. The result follows from choosing $z = 0$. There, $f(0) = v(0, 0) - 0$ and since there is no benefit from a new patent we are in phase 0 so $v(0, 0) = (\pi/r)(1 - exp(-rT)) > 0$.

Finally, effort is increasing through time as $v(0) = v_0$ and

$$\frac{\partial v(t)}{\partial t} = -\frac{2\phi^2\left(\theta^2 - \phi^2\right)e^{\phi(T-t)}}{\lambda^2\left((\theta+\phi)e^{\phi(T-t)} - (\theta-\phi)\right)^2} < 0.$$

B.2 Proof of Propositions 2 and 3

Comparative static results for the value of a new patent are obtained through applications of the implicit function theorem. Before going into the details of the proofs, I describe the general methodology used. Let $f(\alpha, z) = v(0; \alpha, z) - z$ be the construction presented in the proof of Proposition 1 where its dependence on the vector of parameters of the model, α, has been made explicit. By the implicit function theorem there exists a function $V(\alpha)$ implicitly defined by $f(\alpha, V(\alpha)) = 0$ that describes the equilibrium value of having a new patent. Then, the comparative statics for how the value of a new patent, $v(0)$, changes due to a change in parameter α_i is given by

$$\frac{dV(\alpha)}{d\alpha_i} = -\left(\frac{\partial f(\alpha, V(\alpha))}{\partial \alpha_i}\right) \Big/ \left(\frac{\partial f(\alpha, V(\alpha))}{\partial x}\right).$$

From the proof of the existence of equilibrium and the fact that I use the largest fixed point v_0 follows that $\partial f(\alpha, V(\alpha))/\partial x < 0$, so comparative statics are signed by [16]

$$\text{sign}\left(\frac{dV(\alpha)}{d\alpha_i}\right) = \text{sign}\left(\frac{\partial f(\alpha, V(\alpha))}{\partial \alpha_i}\right) = \text{sign}\left(\frac{\partial v(0; \alpha, V(\alpha))}{\partial \alpha_i}\right).$$

For space considerations, only final expressions are shown. Details of the derivation can be found in section XX of the online appendix. For ease on notation, define the positive constant

$$\Gamma = (v_0^2\lambda^2 + 2\pi)/(\theta(e^{\phi T} - 1) + \phi(e^{\phi T} + 1))^2.$$

[15] There may be other fixed points such that $v_0 \leq 0$; however, those do not have an economic meaning and, consequently, are ignored.

[16] For completeness, section XX in the on-line Appendix shows that $\partial f(\alpha, V(\alpha))/\partial x < 0$ analytically.

Comparative static with respect to r: The value of a new patent decreases with an increase in r as:

$$\frac{\partial v(0)}{\partial r} = -\Gamma\left(\left(e^{\phi T}-1\right)^2 + \frac{\theta}{\phi}\left(e^{2\phi T} - 2\phi T e^{\phi T} - 1\right)\right) < 0$$

The only expression that is not clearly signed is $e^{2\phi T} - 2\phi T e^{\phi T} - 1$. I show that this expression is positive for all relevant values of ϕT. Let $y = \phi T$ and define $h(y) = e^{2y} - 2ye^{y} - 1$. Then $h(0) = 0$ and $h'(y) = 2e^{y}(e^{y} - 1 - y) > 0$ which is positive for positive values of y, implying the result.

Comparative static with respect to π: The value of a new patent increases with an increase in π as:

$$\frac{\partial v(0)}{\partial \pi} = 2\Gamma\left(\frac{\left(e^{\phi T}-1\right)^2}{v_0} + \left(e^{2\phi T} - 2\phi T e^{\phi T} - 1\right)\frac{a}{\phi}\right) > 0.$$

Comparative static with respect to b: The partial derivative of $v(0)$ with respect to b is:

$$\frac{\partial v(0)}{\partial b} = \frac{\lambda^2\Gamma}{\phi}\left(\phi v_0\left(e^{\phi T}-1\right)^2 + \left(e^{2\phi T} - 2\phi T e^{\phi T} - 1\right)\left(rv_0 + 3\lambda^2(1-b)v_0^2 + 2\pi(1-2b)\right)\right)$$

and the result follows from the observation that $rv_0 + 3\lambda^2(1-b)v_0^2 + 2\pi(1-2b)$ is guaranteed to be positive whenever $b < 1/2$. [17]

Comparative static with respect to T: The value of a new patent increases with an increase in T as:

$$\frac{\partial v(0)}{\partial T} = 2\Gamma\phi^2 e^{\phi T} > 0.$$

Comparative static with respect to λ: The partial derivative of $v(0)$ with respect to λ is:

$$\frac{\partial v(0)}{\partial \lambda} = -\frac{2\lambda\Gamma}{\phi}\left(\left(e^{\phi T}-1\right)^2\phi(1-b)v_0 + \left(e^{2\phi T} - 2\phi T e^{\phi T} - 1\right)\left(3\lambda^2(1-b)^2 v_0^2 + r(2-b)v_0 - \pi(1-2b(1-b))\right)\right).$$

Again, this expression cannot be signed directly. At the limit, when full forward protection, $b = 1$, is offered, becomes:

$$\frac{\partial v(0)}{\partial \lambda} = 2\lambda\Gamma\left(e^{2\phi T} - 2\phi T e^{\phi T} - 1\right)\left(\frac{\pi - rv_0}{\phi}\right) > 0$$

where the result follows from the fact that getting monopoly profits perpetually and without R&D costs, π/r, must be larger that the equilibrium value of finite patent protection v_0.

[17] It is possible to provide a necessary and sufficient condition for this result. However, it would be as a function of v_0. Hence, not directly in terms of the primitives of the model.

Comparative static with respect to v_0: The partial derivative of $v(0)$ with respect to v_0 is:

$$\frac{\partial v(0)}{\partial v_0} = \frac{\lambda^2 \Gamma}{\phi} \left(\phi b \left(e^{T\phi} - 1 \right)^2 - \left(e^{2T\phi} - 2T\phi e^{T\phi} - 1 \right) \left(3\lambda^2 (1-b)^2 v_0 + (2-b) r \right) \right)$$

which does not have a clear sign. To characterize further take $b = 0$ to get

$$\frac{\partial v(0)}{\partial v_0} = -\lambda^2 \Gamma \frac{3\lambda^2 v_0 + 2r}{\phi} \left(e^{2T\phi} - 2T\phi e^{T\phi} - 1 \right) < 0.$$

which proving the result.

B.3 Proof of Proposition 4

I start describing the general methodology of the proofs. Through an application of the Fundamental Theorem of Calculus, the incumbent's R&D investment at t can be written as

$$x_{i,t} = \lambda (v(0) - v(t)) = -\lambda \int_0^t v'(s)\, ds.$$

Using the observation

$$\frac{dv(t)}{dt} = -\frac{dv(t)}{dT}$$

the comparative statics for parameter α_j (excluding λ) is equal to

$$\frac{dx_{i,t}}{d\alpha_j} = \lambda \int_0^t \frac{d^2 v(s)}{d\alpha_j dT} ds.$$

Therefore, if the sign of $d^2 v(t)/d\alpha_j dT$ is constant through t, it is a sufficient descriptive to sign R&D comparative statics.

B.4 Proof of Proposition 5

The total derivative with respect to patent length is

$$\frac{dx_{i,t}}{dT} = \lambda \left(\frac{dv_0}{dT} - \left(\frac{\partial v(t)}{\partial T} + \frac{\partial v(t)}{\partial v_0} \frac{dv_0}{dT} \right) \right),$$

when the change in policy is grandfathered to the next invention, there is no direct effect, i.e. $\partial v(t)/\partial T = 0$ and the derivative becomes.

$$\frac{dx_{i,t}}{dT} = \lambda \frac{dv_0}{dT} \left(1 - \frac{\partial v(t)}{\partial v_0} \right).$$

From Proposition 2 we know that $dv(t)/dT > 0$ and the result follows from the observation that, by construction, $\partial v(t)/\partial v_0 < 1$.

B.5 Proof of Proposition 7

I start by showing that the limiting value of a patent is given by equation (**??**). Taking the limit of (6) when T goes to infinity delivers

$$v_\infty = \lim_{T\to\infty} v\left(t\right) = \frac{2\pi + \left(\lambda v_\infty\right)^2}{\theta + \phi}.$$

Solving this expression for v_∞ delivers a unique positive solution. When $b < 1$ the solution is

$$v_\infty = \frac{1}{2\lambda^2\left(1-b\right)^2}\left(-r + \sqrt{r^2 + 4\pi\lambda^2\left(1-b\right)^2}\right)$$

and when $b = 1$ the solution is $v_\infty = \pi/r$, which are the expressions in the proposition, proving the second statement. The incumbent investments converging to zero is just consequence of $\lim_{T\to\infty} v(t) = v_\infty$ for all t.

C Strategic Entrants

In this appendix I derive the ODE describing how the value of a patent, $v\left(t\right)$, evolves through out its life and the value of an being an entrant that faces an incumbent that posses a patent at t, $w\left(t\right)$. To start deriving the value of competing after patent protection expires to then compute the values under protection.

C.1 Continuation value of competing after patent protection expires

After patent protection expires, the game becomes stationary and so will be the firms investmet rates, thus each firm solves

$$\begin{aligned} c_j &\equiv \max_{x_j} \int_0^\infty \left(\lambda x_j v_0 - \frac{\left(x_j\right)^2}{2} + \lambda x_{-j} w_0\right) e^{-\left(\lambda\left(x_j + x_{-j}\right) + r\right)t} ds \\ &= \max_{x_j} \frac{2\lambda x_j v_0 - x_j^2 + 2\lambda x_{-j} w_0}{2\left(\lambda\left(x_j + x_{-j}\right) + r\right)} \end{aligned}$$

where $x_{-j} = \sum_{i\neq j} x_i$, v_0 is the value of a new patent and w_0 the value of being an entrant when a new patent was issued. Taking the first order condition and imposing symmetry delivers two posible solutions, but only one solution delivers a positive investment rate of

$$x^* = \frac{1}{\lambda\left(2m+1\right)}\left(\lambda^2 m\left(v_0 - w_0\right) - r + \sqrt{\left(r + \lambda^2 m\left(v_0 - w_0\right)\right)^2 + 2r\lambda^2\left(v_0 + 2mw_0\right)}\right)$$

which in turn implies that the value of competing in a race with no protection is given by

$$c = \frac{1}{\lambda^2\left(2m+1\right)}\left(r + \lambda^2\left(v_0 + m\left(v_0 + w_0\right)\right) - \sqrt{\left(r + \lambda^2 m\left(v_0 - w_0\right)\right)^2 + 2r\lambda^2\left(v_0 + 2mw_0\right)}\right).$$

C.2 Value function during patent protection

Let $\hat{x}_{n,s} = \sum_{j=1}^{m} x_{n_i,s}$, then the value of being an incumbent at that posess a patent at t is

$$v(t) = \int_t^T \left(\pi - \frac{x_{i,s}^2}{2} + \lambda x_{i,s} v_0 + \lambda \hat{x}_{n,s} \left(w_0 + b \left(v(s) - c \right) \right) \right) e^{-z_{0,s}} e^{-r(s-t)} ds$$

This version of compensantion is as follows: What is the difference between the value of a patent today, to the value of patent protection terminating today. and the value for entrant j

$$w_j(t) = \int_t^T \left(-\frac{x_{n_j,s}^2}{2} + \lambda x_{n_j,s} \left(v_0 - b \left(v(s) - c \right) \right) + \lambda \left(x_{i,s} + \hat{x}_{-n_j,s} \right) w_0 \right) e^{-z_{0,s}} e^{-r(s-t)} ds.$$

Appliying the Principle of Optimality to equations (X) and (Y) I, respectivelly obtain

$$\begin{aligned} 0 &= \max_{x_{i,t}} \left\{ v'(t) + \pi - \frac{x_{i,t}^2}{2} + \lambda x_{i,t} \left(v_0 - v(t) \right) + \lambda \hat{x}_{n,t} \left(w_0 - bc - (1-b) v(t) \right) - v(t) r \right\} \\ 0 &= \max_{x_{n_j,t}} \left\{ w_j'(t) - \frac{x_{n_j,t}^2}{2} + \lambda x_{n_j,t} \left(v_0 - b \left(v(t) - c \right) - w_j(t) \right) + \lambda \left(x_{i,t} + \hat{x}_{-n_j,t} \right) \left(w_0 - w_j(t) \right) - w(t) r \right\} \end{aligned}$$

The first order conditions are

$$x_{i,t}^* = \lambda \left(v_0 - v(t) \right) \quad , \quad x_{n_j,t}^* = \lambda \left(v_0 - w_j(t) - b \left(v(t) - c \right) \right)$$

which together with symmetry ($x_{n_j,t} = x_{n,t}$ and $w_j(t) = w(t)$ for all j) imply the following system of differential equiations

$$\begin{aligned} 0 &= v'(t) + \alpha_0 v(t)^2 - \alpha_1 v(t) + \lambda^2 m (1-b) w(t) v(t) - m\lambda^2 (w - bc) w(t) + \alpha_2 \\ 0 &= w'(t) + \frac{b^2 \lambda^2}{2} v(t)^2 - \alpha_3 v(t) + \lambda^2 (1 + mb) w(t) v(t) - \alpha_4 w(t) + \lambda^2 \left(m - \frac{1}{2} \right) w(t)^2 + \alpha_5 \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= \lambda^2 \left(\frac{1}{2} + mb(1-b) \right) \\ \alpha_1 &= \left(r + \lambda^2 v \left(1 + m(1-b) \right) + \lambda^2 m (1-2b) bc + \lambda^2 mbw \right) \\ \alpha_2 &= \pi + \frac{1}{2} v^2 \lambda^2 + m\lambda^2 \left((v + bc)(w - bc) \right), \\ \alpha_3 &= \lambda^2 \left(w + b \left(v + (m-1) w \right) + b^2 c \right) \\ \alpha_4 &= \left(r + \lambda^2 \left(v - w + m \left(v + w + bc \right) \right) \right) \\ \alpha_5 &= \lambda^2 \left(\frac{v^2 + b^2 c^2}{2} + bc(v - w) + mw(v + bc) \right) \end{aligned}$$

C.3 Free Entry and Convergence of the Basic Model

In this subsection I show that a model under free entry plus a particular assumption in the entrant's arrival process converges to the basic model presented in the introduction. There is two arrival process, one where for the incumbent which delivers innoations at a rate $\lambda x_{i,t}$ and one for the entrants, that delivers innovation at a rate $\lambda \hat{x}_{n,t}$ where $\hat{x}_{n,t} = \sum_{j=1}^{m} x_{n,,t}$. Then, the successful entrant is chosen with uniform probability, $1/m$, among all the entrants. Under this assumption equation (X) remains the same, but equation (Y) changes to

$$w_j(t) = \int_t^T \left(-\frac{x_{n_j,s}^2}{2} + \frac{\lambda}{m} \left(x_{n_j,t} + \hat{x}_{-n_j,s} \right) \left(v_0 - b\left(v(s) - c \right) \right) + \lambda \left(x_{i,s} + \hat{x}_{-n_j,s} \right) w_0 \right) e^{-z_{0,s}} e^{-r(s-t)} ds.$$

Appliying the principle of optimality we obtain the following HJB equation

$$\begin{aligned} w_j(t) &= \max_{x_{n_j,t}} \left\{ \left(-\frac{x_{n_j,s}^2}{2} + \frac{\lambda}{m} \left(x_{n_j,t} + \hat{x}_{-n_j,t} \right) \left(v_0 - b\left(v(t) - c \right) \right) + \lambda \left(x_{i,t} + \hat{x}_{-n_j,t} \right) w_0 \right) dt + (1 - (r + x_{i,t} + \hat{x}_n \right. \\ 0 &= \max_{x_{n_j,t}} \left\{ \left(-\frac{x_{n_j,s}^2}{2} + \frac{\lambda}{m} \left(x_{n_j,t} + \hat{x}_{-n_j,t} \right) \left(v_0 - b\left(v(t) - c \right) \right) - \lambda x_{n_j,t} w(t) + \lambda \left(x_{i,t} + \hat{x}_{-n_j,t} \right) \left(w_0 - w(t) \right) \right) \right. \end{aligned}$$

FOC

$$x_{n_j,t} = \frac{\lambda}{m} \left(v_0 - b\left(v(t) - c \right) \right) - \lambda w_j(t)$$

thus with symmetry

$$\begin{aligned} x_{n,t} &= \frac{\lambda}{m} \left(v_0 - b\left(v(t) - c \right) \right) - \lambda w(t) \\ \hat{x}_{n,t} &= \lambda \left(v_0 - b\left(v(t) - c \right) \right) - \lambda m w(t) \end{aligned}$$

converging to the model.